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VALUATION OF ASSET LEASING CONTRACTS*

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This paper describes the relation among a variety of asset leasing contracts, including: (1) cancellable operating leases; (2) leases which grant the lessee an option to extend the life of the lease; (3) leases that grant the lessee an option to purchase the leased asset at a fixed price at the maturity date of the lease; (4) leases that grant the lessee the right to purchase the leased asset at its 'fair market value' at the maturity date of the lease; (5) leases that grant an option to the lessee to purchase the leased asset at a prespecified price anytime during the life of the lease; (6) leases that require the lessee to purchase the leased asset at a fixed price at the maturity date of the lease; and (7) leases that contain non-cancellation provisions. The paper uses a compound option pricing framework to develop a general model for valuing (or evaluating) each of the types of leasing contracts. Numerical examples are presented to illustrate the effect of the various elements of a leasing contract — including cancellation risk and residual value risk — on equilibrium rental payments.

1. Introduction

The literature of finance has devoted considerable attention to the analysis of asset leasing contracts.[1] For the most part, this literature has focused on the valuation of financial leases. The key distinguishing feature of a financial lease is that it is non-cancellable during the life of the contract. At the opposite end of the spectrum from the financial lease is the operating lease. The key distinguishing feature of the operating lease is that it may be cancelled at any time during the life of the contract.[2]

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[1]See, among others, Bower (1973), Brennan and Kraus (1977), Franks and Hodges (1978), Lewellen et al. (1976), Myers et al. (1976), and Schall (1974).

[2]Testimony to the importance of the potential for premature termination in the valuation of lease contracts is given by the recent spectacular failures of two international leasing firms, O.P.M. Leasing Services and Itel Corporation. The failure of both firms has been widely

Both the financial lease and the operating lease typically have a fixed maturity, call for equal periodic rental payments, and require that the leased asset be returned to the lessor at the maturity date of the contract. But these are not the only provisions that can be written into a lease. For example, there exist: (1) leases which grant the lessee the right to extend the life of the lease beyond its original maturity date; (2) leases which grant the lessee the right to purchase the leased asset at a fixed price at the maturity date of the contract; (3) leases which grant the lessee the right to purchase the leased asset at its fair market value at the maturity date of the contract; (4) leases which grant the lessee the right to purchase the leased asset at any time during the life of the contract (usually at contractually fixed prices that vary through time); (5) leases that require the lessee to purchase the asset at the maturity date of the contract (this type of lease is generally called an open-end lease); and (6) leases that specify a period over which the lease is non-cancellable followed by a period over which the lease may be cancelled at any time. (This latter lease can be thought of as a financial lease coupled with an operating lease.)

The purpose of this paper is to present a general model for valuing (or evaluating) asset leasing contracts. The approach that we adopt characterizes leasing contracts as compound options. In section 2, we use option terminology to describe what we shall call the standard operating lease. With this type of lease, the lessee receives the right to use the leased asset over a specified period of time in exchange for the payment of contractually fixed periodic rentals. At the maturity of the lease or upon termination of the lease by the lessee, the use value of the asset reverts to the lessor. Additionally, we describe the way in which the other types of leases enumerated above are related to the standard operating lease. Embedded in this discussion are the assumptions that capital markets are frictionless and that investors are rational and have positive marginal utility for wealth.

In section 3, we develop a specific model for valuing the standard operating lease. We then show the way in which this model can be easily modified to value the various other leases listed above, including the financial lease. Thus, the valuation model has general applicability.

In section 4, we consider the comparative statics of the valuation model and use the model to evaluate some hypothetical leases. These numerical examples are intended to demonstrate the way in which the model can be used by lessees to make leasing decisions and by lessors to determine required rental payments.

The final section of the paper is a summary.

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attributed to the premature cancellation of leases for computer equipment. Massive terminations of leases for computer equipment occurred in 1980 and 1981 when the introduction of more efficient computers made older ones obsolete.

2. A characterization of asset leasing contracts

2.1. The standard operating lease

According to the terms of the standard operating lease, the lessee receives the right to use the leased asset over a specified period of time in exchange for predetermined rental payments. The first rental payment is made when the lease is initiated. Subsequent payments are due at periodic dates in the future.

The lease that we will consider here calls for n rental payments of $\bar{L}$ each. The n rental payments are due at equal intervals prior to the maturity date of the contract, T. According to the lease, the lessee receives all service flows generated by the asset over the time interval between rental payments. If the lessee decides not to make a specified lease payment, the lessor has a claim to the residual value of the asset only. Upon the omission of a lease payment, the lessor may assign the right to use the asset to another lessee.

The standard operating lease can be viewed as a compound option. Specifically, with this type of lease, the last rental payment purchases the use of the leased asset over the time remaining until the contract expires. Payment of the next to the last rental payment purchases the use of the asset over the time interval until the next lease payment is due and it purchases an option to make the last rental payment. Payment of the third from the last rental payment purchases the use of the asset over the period until the next to the last payment is due and it purchases an option to make the next to the last rental payment. But the next to the last rental payment also purchases an option. Thus, all rental payments prior to the last one purchase compound options with exercise prices of $\bar{L}$ each. Based on this line of reasoning, in the third section of the paper we employ Rubinstein's (1976) method for valuing risky income streams and Geske's (1977) method for valuing compound options to derive a model for valuing asset leasing contracts.[3]

2.2. Lease with option to extend the maturity date of the contract

Some lease contracts — generally called 'renewable' leases — grant the lessee the right to extend the maturity date of the lease for a specified number of periods beyond the original maturity date. That is, a lease with maturity date T may be extended to mature at date $T+K$ where K is the number of additional rental payments permitted under the extension. The periodic required rental payments typically are the same under the extension as they were under the original lease.

[3]Smith (1979) and Copeland and Weston (1982) also recognize that leases can be evaluated in an option pricing framework. Like us, Smith analyzes the lease as a call option. Contrarily, Copeland and Weston analyze the lease as a put option.

The relationship between the standard operating lease and the extendable — or renewable — lease is straightforward. Specifically, an n-payment standard operating lease with maturity date T and with an option to extend the lease until date $T+K$ by making K additional lease payments is equivalent to an $(n+K)$-payment standard operating lease with maturity date $T+K$. In either case, the lessee may cancel the lease at any time between the initiation of the lease and date $T+K$. Thus, the equilibrium rental payment for a standard operating lease with maturity date $T+K$ is the same as the rental required of an extendable lease that may be extended to date $T+K$.

2.3. Lease with option to purchase at maturity

Frequently, operating leases grant the lessee an option to purchase the leased asset at a fixed price, P_T, at the maturity date of contract. This lease also can be viewed as a compound option in which the first n options have exercise prices of $\bar{L}$ each and in which the $n+1$ option has an exercise price of P_T. Payment of the first n exercise prices purchases use of the asset over the subsequent period and it purchases the remaining options. Payment of the last exercise price purchases the residual value of the asset at the maturity of the contract. Thus, to value this lease it is only necessary to incorporate an additional option with exercise price P_T. With all else equal, the effect of this additional option is to increase the periodic rental payments.

In many cases leases grant the lessee an option to purchase the leased asset at its 'fair market value' at the maturity date of the contract. However, the lessee can purchase the asset at its market price at maturity of the lease whether or not the contract contains such an option. Thus, an option to purchase the asset at its fair market price is valueless and the equilibrium rental payments will be the same whether or not the lease contract contains such an option.

2.4. Lease with option to purchase at any time

In some cases lease contracts permit the lessee to purchase the asset at any time during the life of the lease. Generally the contract specifies that the purchase price will decline after each lease payment by the amount of the payment. Although the exercise price varies over time, it has a fixed value at each point in time.

In a fashion analogous to Merton's (1973) proof that a rational investor will never exercise an American call option prior to maturity on a non-dividend paying stock, it can be shown that a rational lessee will never exercise the option to purchase a leased asset prior to maturity when the purchase price declines by the amount of the contractual lease payment on the date that each rental payment is due. This result comes about because the option holder (i.e., the lessee) receives the service flows generated by the

asset (i.e., the option is dividend protected).[4] On the one hand, the lease with a purchase option enables the lessee to use the asset without bearing the 'downside' risk in residual value. On the other hand, the lessee benefits when the residual value is greater than the exercise price at maturity. With premature exercise of the purchase option, the lessee would forfeit this 'insurance' against the downside risk in residual value. With no gain from premature exercise and with the possibility of loss, the lessee would never exercise the purchase option prior to maturity.

Thus, if $\hat{P}_T$ is the *final* purchase price under an n-payment lease which contains an option to purchase the leased asset at *any time* during the life of the lease at a price that declines at the date of each rental payment by the amount of the rental payment, then from the analysis above, the competitive rental payments (and, therefore, the value) of this lease will be identical to those of an n-payment lease with an option to purchase the asset *only at maturity* at the fixed price $P_T = \hat{P}_T$.

2.5. Lease with an asset purchase requirement

Under an 'open-end' lease (alternatively known as a conditional sales lease) the lessee is *required* to purchase the leased asset at a fixed price at the maturity date of the lease. The lessor is, of course, obligated to sell the asset at the same fixed price. An additional provision of the open-end lease is that all (past and future) lease payments, including the fixed purchase price of the asset, are due upon default of any provisions of the lease contract. Furthermore, the lessee is required to pledge sufficient security to collateralize these provisions of the lease. Under these conditions a rational lessee will never terminate an open-end lease prior to maturity. The reasoning behind this result is as follows: Under an open-end lease the lessee is required to make the lease payments and the asset purchase payment through time. If the lessee defaults on the lease, the payments are due immediately. Assuming a positive time value of money, the lessee will be worse off making the payments sooner rather than later. Thus, the payments on an open-end lease will be made as scheduled.[5] Under these conditions, the open-end lease is equivalent to a standard operating lease with a purchase option on which each of the options (including the purchase option) will be exercised with certainty.

2.6. Lease with non-cancellation period

Finally, some operating leases specify non-cancellation periods during which the lessee may not terminate the lease. Following the non-cancellation

[4]A formal proof of this proposition is available upon request.
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period the contract becomes a standard (cancellable) operating lease. As with the lease with an asset purchase requirement, the lessee is required to pledge sufficient collateral so as to guarantee the lease payments during the non-cancellation period.

In the extreme, a lease may be non-cancellable during the entire life of the contract. But, this brings us full circle to the financial lease. As with the open-end lease, the omission of any lease payment on the financial lease makes all the lease payments due and payable immediately. Thus, if the collateral is sufficiently valuable to fully guarantee the lease payments, the financial lease can be viewed as a standard operating lease on which each of the options will be exercised with certainty.[6]

Financial leases may also contain provisions that grant the lessee the right to purchase the asset: (1) at the maturity date of the contract at a fixed price; (2) at the maturity date of the contract at its fair market value; and (3) at any time during the life of the contract at fixed prices that vary through time. Because a non-cancellable financial lease is equivalent to a standard operating lease on which each of the lease payments will be made with certainty, each of the results in sections 2.3 and 2.4 derived for the standard operating lease will apply to the financial lease as well.

3. A model for valuing asset leasing contracts

3.1. Valuing the standard operating lease

In this section we present a model for valuing the standard operating lease as a compound option. In the following sections we show how the model can be adapted to value the other leases described above. To develop the model, we employ the following set of assumptions:

(A.1) Single-price law of markets: All securities or portfolios of securities with identical payoffs sell at the same price.

(A.2) Non-satiation: The larger its payoff in any state, the greater the current price of a security.

(A.3) Perfect, competitive and Pareto-efficient capital markets: The capital market is perfectly competitive with no transactions costs, no taxes, and equal access to information by all participants.

[6]Some elaboration may be appropriate here. In most cases the corporation pledges its 'full faith and credit' to guarantee the non-cancellable financial lease. That is, *all* of the firm's assets are used to collateralize the lease, but that is not necessary. Some smaller (or larger) set of assets could be pledged. If the value of the firm or the pledged assets is sufficiently high, the lease will be 'default-free'. If the value of the pledged assets is not sufficiently high, there will be some probability that the lease payments will not be made. In that case, the financial lease would still be valued as a compound option in which there is some probability that the various options will not be exercised. The same is true of the 'open-end' lease. We return to this point in footnote 9.

(A.4) Lognormality of returns: The return on the underlying (leased) asset and the return on aggregate wealth are jointly lognormally distributed.

(A.5) Weak aggregation: The conditions for weak aggregation are met so that securities are priced as though all investors have the same characteristics as the average investor.

(A.6) Constant proportional risk aversion: The average investor exhibits constant proportional risk aversion (CPRA).

(A.7) The distribution of the rate of economic depreciation of the leased asset is stationary over time.

(A.8) The risk-free rate of interest is constant over time.

Assumptions (A.1)–(A.4) are commonly employed in the development of financial models. Rubinstein (1976) has used Assumptions (A.5) and (A.6) in conjunction with Assumptions (A.1)–(A.4) to develop a method for valuing risky income streams with discrete payoffs. Because lease payments typically are due at discrete time intervals we will employ Rubinstein's techniques to value the lease contract. Assumptions (A.7) and (A.8) are made primarily for convenience.

The standard operating lease offers the lessee the opportunity to make n periodic lease payments to purchase the use of the asset until time T. For the convenience of presentation and notation, we assume that each lease payment covers a single time period.

As with the solution to other option valuation problems, the key to valuing the standard operating lease is the determination of appropriate boundary conditions. To establish the boundary conditions for the standard operating lease consider the decision confronting the lessee at the date on which the last rental payment is due. At that time, payment of $\bar{L}$ purchases the use of the asset over the time interval from $T-1$ to T. Let L_{T-1} be the competitively determined equilibrium market rental at time $T-1$ to purchase the use of an identical asset over the period from $T-1$ to T. If $L_{T-1} \geqq \bar{L}$, the lessee will make the rental payment. If $L_{T-1} < \bar{L}$ the lessee will cancel the lease, choosing instead to rent the identical asset at the market determined rental L_{T-1}. Thus, the lessee's decision at time $T-1$ involves a comparison of the contractual rental $\bar{L}$ with the current market rental to rent the asset for the single time interval $T-1$ to T.

Miller and Upton (1976) demonstrate that the rental on a single-period lease will be just sufficient to compensate the lessor for the opportunity cost of capital invested in the leased asset over the time period covered by the lease plus the expected loss in the value of the asset due to economic depreciation over the same time period. Given the analysis of Miller and

Upton, Assumptions (A.1)–(A.8), and the valuation techniques of Rubinstein (1976), the equilibrium rental at time $T-1$ for a single-period lease that extends over the interval from $T-1$ to T is

$$L_{T-1} = A_{T-1} - [(1 - \mathrm{E}(\tilde{d}))/(1 + r_f)]\, \mathrm{e}^{\sigma_{ly}} A_{T-1}, \tag{1}$$

where A_{T-1} is the market value of the leased asset at time $T-1$, $\mathrm{E}(\tilde{d})$ is the expected rate of economic depreciation of the asset, σ_{ly} is the covariance between the logarithm of one minus the rate of economic depreciation and the 'market factor' y, and r_f is the risk-free rate of interest.[7] Note that the rate of economic depreciation is defined in market value terms as $(A_{T-1} - A_T)/A_{T-1}$.

From above, the boundary condition for the standard operating lease at time $T-1$ is $L_{T-1} \geqq \bar{L}$. If this condition is satisfied at $T-1$, the lease payment will be made. If not, the lessee will terminate the lease.

[7]The competitive market rental for the single-payment lease is derived as follows. From Miller and Upton (1976), the single-payment rental is the difference between the current asset price (A_{T-1}) and the discounted value of the asset's residual value at the end of the period. We will denote the present value of the asset's residual value as S_{T-1}, so that the single-payment rental can be represented as

$$L_{T-1} = A_{T-1} - S_{T-1}. \tag{i}$$

Using Rubinstein's (1976) technique for valuing risky income streams, we can write the present value of the future residual value of the asset as

$$S_{T-1} = \mathrm{E}(\tilde{A}_T Z'_T), \tag{ii}$$

where $Z'_T = R_{mT}^{-b} R_{fT}^{-1} / \mathrm{E}(R_{mT}^{-b})$, R_{mT} is the return on aggregate wealth from $T-1$ to T, b is the measure of CPRA and R_{fT}^{-1} is the current price of a risk-free security maturing at T. From Assumption (A.8), $R_{fT}^{-1} = (1 + r_f)^{-1}$ for all T. Note also that $\mathrm{E}(Z'_T) = R_{fT}^{-1}$. Then define $\tilde{\delta}_T \equiv \tilde{A}_T / A_{T-1}$, and from Assumption (A.7), $\mathrm{E}(\tilde{\delta}_T) = [1 - \mathrm{E}(\tilde{d})]$ for all T. We can now write S_{T-1} as

$$S_{T-1} = A_{T-1} \mathrm{E}(\tilde{\delta}_T Z'_T). \tag{iii}$$

Then define $l \equiv \ln(\tilde{\delta}_T)$ and $y \equiv \ln(Z'_T)$ so that

$$S_{T-1} = A_{T-1} \mathrm{E}(\mathrm{e}^l \cdot \mathrm{e}^y). \tag{iv}$$

Assumption (A.4) implies that $\tilde{\delta}_T$ and Z'_T are jointly lognormal. Thus, l and y are jointly normal and

$$\begin{aligned} S_{T-1} &= A_{T-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^l \cdot \mathrm{e}^y f(l, y)\, \mathrm{d}l\, \mathrm{d}y, \\ &= A_{T-1} \exp[\mu_l + \mu_y + 1/2\sigma_l^2 + \sigma_{ly} + 1/2\sigma_y^2], \\ &= A_{T-1} \cdot \mathrm{E}(\tilde{\delta}_T) \cdot \mathrm{E}(Z'_T) \cdot \mathrm{e}^{\sigma_{ly}}. \end{aligned} \tag{v}$$

Finally, from (i) and (v), the rental payment for the single-payment lease is

$$L_{T-1} = A_{T-1} - [(1 - \mathrm{E}(\tilde{d}))/(1 + r_f)]\, \mathrm{e}^{\sigma_{ly}} A_{T-1}.$$

Because the relationship expressed in (1) will apply at each point in time to a single-payment lease, it can be used to solve recursively for the equilibrium rental on a standard operating lease at dates prior to $T-1$. Consider the decision of the lessee at time $T-2$. The lessee must choose between making the lease payment $\bar{L}$ and entering into a two-payment lease for an identical asset. The two-payment lease purchases the use of the asset over the single period from $T-2$ to $T-1$ and a call option on the use of the asset from $T-1$ to T. The exercise price of the call option is $\bar{L}$. Thus, the two-payment lease encompasses a single-payment lease plus a call option.

For convenience, let $\lambda=[(1-\mathrm{E}(\tilde{d}))/(1+r_f)]\,\mathrm{e}^{\sigma_{ly}}$. Then from (1) the market-determined rental on a single-payment lease that rents the asset over the period from $T-2$ to $T-1$ is

$$L_{T-2}=(1-\lambda)A_{T-2}. \tag{2}$$

Given Assumptions (A.1)–(A.8) and Rubinstein (1976) the value of the option to use the asset from $T-1$ to T is

$$C_{T-1}=\mathrm{E}[(L_{T-1}-\bar{L})Z'_{T-1}\,|\,A_{T-1}>\bar{A}_{T-1}], \tag{3}$$

where Z'_{T-1} is the price of an Arrow–Debreu primitive security divided by a probability measure, E is the expectations operator and $\bar{A}_{T-1}$ is the value of A_{T-1} such that $L_{T-1}=\bar{L}$. The condition $A_{T-1}>\bar{A}_{T-1}$ occurs if and only if $L_{T-1}>\bar{L}$; thus $\bar{A}_{T-1}$ is the asset value above which the lessee will choose to make the lease payment at $T-1$ to extend the life of the lease to the maturity date T.

Let $L^2_{T-2}(\bar{L})$ represent the market-determined rental at time $T-2$ for a two-payment lease with two fixed lease payments of $\bar{L}$ each period. Combining (2) and (3), the equilibrium rental at time $T-2$ for a two-payment standard operating lease that matures at date T is

$$L^2_{T-2}(\bar{L})=L_{T-2}+C_{T-1}. \tag{4}$$

Thus the boundary condition for the multiple-payment lease at time $T-2$ is $L^2_{T-2}(\bar{L})\geqq\bar{L}$.

Proceeding recursively, consider the lessee's decision at date $T-3$. At this point payment of $\bar{L}$ purchases the use of the asset from $T-3$ to $T-2$ and it purchases an option to make the next lease payment at date $T-2$. But the lease payment at $T-2$ also contains an option to extend the lease at time $T-1$. If we define $L^3_{T-3}(\bar{L})$ as the equilibrium market rental at time $T-3$ for a three-period lease that purchases the use of the asset from $T-3$ to $T-2$ and purchases a compound option with exercise price of $\bar{L}$ to extend the lease at $T-2$ and again at $T-1$, the boundary condition for the three-period lease at

$T-3$ is $L^3_{T-3}(\bar{L}) \geqq \bar{L}$. From eqs. (2) and (3) we get

$$L^3_{T-3}(\bar{L}) = (1-\lambda)A_{T-3} + \mathrm{E}[(L_{T-2} - \bar{L})Z'_{T-2} \,|\, A_{T-2} > \bar{A}_{T-2}], \tag{5}$$

where $\bar{A}_{T-2}$ is defined analogously to $\bar{A}_{T-1}$. Because the solution for L_{T-2} also contains an option, eq. (5) contains a compound option.

This procedure for determining the boundary conditions can be repeated at $T-4$, $T-5$, and so on until the date at which the n-payment standard operating lease is initiated, $T-n$. Using Rubinstein's (1976) and Geske's (1977) solution techniques for valuing risky cash flows, we can derive closed-form solutions to eqs. (3) and (5). The general solution for an n-period operating lease is as follows:

Valuation of the standard operating lease: Given Assumptions (A.1)–(A.8) the *equilibrium lease payment* for an n-payment standard operating lease with the first payment due at $T-n$ (i.e., at time 0) and with future rental payments of L^* each due at equal time intervals in the future until time $T-1$ and with the maturity of the contract occurring at T is

$$L^n_0(L^*) = L^* = \sum_{i=0}^{n-1} \lambda^i(1-\lambda)A_0 \cdot N_i(h_i + \sigma\sqrt{i}; \{\rho\})$$

$$- L^* \sum_{i=1}^{n-1} R_f{}^{-i} \cdot N_i(h_i; \{\rho\}), \tag{6}$$

where

$$h_i = (\ln(\lambda^i A_0/\bar{A}_i) + (\ln R_f - \sigma^2/2)i)/\sigma\sqrt{i}.$$

$L^{n-i}_i(L^*)$ is the market rental on an equivalent lease contract at time i, $\bar{A}_i$ is the value of A_i such that $L^{n-i}_i(L^*) = L^*$, A_0 is the current market value of the asset, σ^2 is the variance of the logarithm of the rate of change in the value of the leased asset, hereafter for simplicity we refer to this as the variance rate, and $N_i(\cdot)$ is the i-dimensional multivariate normal distribution function.

The multivariate normal distribution function appearing in eq. (6) is computed as follows:

$$N_n(h_i) = \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \cdots \int_{-\infty}^{h_n} f(x_1, x_2, \ldots, x_n)\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

In turn, $f(x)$, where x is the vector $(x_1, x_2, \ldots, x_n)$ is the multivariate normal density function

$$f(x) = (1/(2\pi)^{n/2}|\Sigma|^{\frac{1}{2}}) \exp[-\tfrac{1}{2}x'\Sigma^{-1}x],$$

where

$$\Sigma = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & & & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix}.$$

Because the lease payments are equally spaced, Σ is symmetric. Furthermore, the time correlation between any two random variables x_{t_i} and x_{t_j} observed from present time 0, where $t_i < t_j$, is equal to the square root of the smaller time divided by the larger time interval: $\rho_{ij} = \sqrt{t_i/t_j}$. If $t_i = i$ and $t_j = j$, then $\rho_{ij} = \rho_{ji} = \sqrt{i/j}$ for all $i < j$.

Because the proof of eq. (6) is analogous to Geske's (1977) proof of the valuation equation for risky coupon bonds we will merely sketch the proof here. The proof of eq. (6) employs Rubinstein's (1976) 'transformed' market parameter Z' and the conditional expectations operator to derive a multiple integral equation for the value of the equilibrium lease payment. Then, given the lognormality and stationarity assumptions a second integral equation is derived (see footnote 7). The limits of the integrals are a function of the boundary conditions for the lease payment, i.e., the $\bar{A}$'s. The closed-form valuation equation is then derived by means of a change of variables and by transforming the integrals to multivariate normal distribution functions.

Because the major emphasis of the extant literature has been the normative aspects of asset leasing, consideration of the valuation of lease contracts immediately raises the question of the way in which the model can be used by firms in deciding whether to accept a specific lease. Ignoring for the moment that the model developed here assumes that lessees will be indifferent to leasing the asset and acquiring the use value of the asset through some other financing arrangement, implementation of the model requires knowledge of the asset's current market price and the risk-free rate of interest along with an estimate of the asset's expected rate of economic depreciation, the covariance between the logarithm of one minus the rate of economic depreciation and the 'market factor', and the variance rate of the asset's market value through time.

For assets that have reasonably well developed second-hand markets, such as most types of computer equipment, aircraft, railroad rolling stock, automobile fleets, office equipment, appliances, and many other durables that comprise the bulk of the leasing market, estimation of the asset's expected rate of economic depreciation and its variance rate through time should be straightforward. Estimation of the required covariance term necessitates some additional assumptions about the appropriate market factor, though in many circumstances it may be sufficient to use a representative stock market index

for this purpose in a fashion analogous to the computation of securities' 'betas'. With these estimates and a specification of the maturity of the lease and the dates upon which the periodic rental payments are due, a lessee could solve for the equilibrium lease payment and compare that with the required contractual lease payment. If the contractual lease payment is less than the estimated equilibrium lease payment, the lease is a 'good deal'. On the other side of the coin, the model could also be used by lessors to determine appropriate rental payments to require of potential lessees. Note, of course, that with a standard operating lease, the terms of the contract are independent of the specific lessee, but depend only on the characteristics of the asset in question (assuming that the expected rate of economic depreciation of the asset is independent of the asset user).

Eq. (6) can also be used to compute the net advantage to leasing (NAL) that has been the focus of the leasing literature. Merely shift L^* from the left-hand side to the right-hand side of the equation, substitute the contractually specified lease payments, $\bar{L}$, for L^* and compute the net advantage to leasing as

$$NAL = \sum_{i=0}^{n-1} \lambda^i (1-\lambda) A_0 \cdot N_i(h_i + \sigma\sqrt{i}; \{\rho\}) - \bar{L} \cdot \sum_{i=0}^{n-1} R_f^{-i} \cdot N_i(h_i; \{\rho\}). \qquad (7)$$

Then, given the terms of the lease contract ($\bar{L}$, n, A_0), characteristics of the leased asset (σ^2, $\mathrm{E}(\tilde{d})$), and market factors (R_f, σ_{ty}), eq. (7) can be used by the lessee to determine the desirability of the lease.

3.2. Valuing the lease with option to extend the maturity of the contract

As discussed in section 2.2, some operating leases grant the lessee the right to extend the original maturity date of the lease, T, to a new maturity date, $T+K$, with the payment of K additional periodic-rental payments. From the discussion in section 2.2, eqs. (6) and (7) may be used to value any standard operating lease with an extension — or renewable — option merely by setting the maturity date of the lease equal to the last extension date of the extendable lease, $T+K$.

3.3. Valuing the lease with option to purchase at maturity

Valuation of a lease that gives the lessee the option to purchase the leased asset at a fixed price, P_T, at the maturity date of the lease, time T, requires only that the boundary conditions be altered to incorporate the additional option. Thus, at T the lessee will purchase the asset if $A_T \geqq P_T$. Then at date $T-1$ the lessee will make the lease payment $\bar{L}$ if $L^1_{T-2}(P_T) > \bar{L}$, where $L^1_{T-2}(P_T)$ is the market-determined lease payment required for a single

payment lease extending from $T-1$ to T that includes an option to purchase the asset at time T for a fixed price of P_T and where $\bar{L}$ is the contractual rental payment specified in the n-payment lease with an option to purchase the asset at maturity. From eqs. (2) and (3) the solution for the single payment lease with a purchase option is

$$L^1_{T-2}(P_T)=(1-\lambda)A_{T-1}+\mathrm{E}[A_T-P_T)Z'_T\,|\,A_T>\bar{\bar{A}}_T], \tag{8}$$

where $\bar{\bar{A}}_T$ is the value of A_T such that $\bar{\bar{A}}_T=P_T$ and all other terms are as defined above. Thus, the required market rental for a single payment lease with an option to purchase is merely the market rental for a single-payment lease plus the value of the purchase option at $T-1$. Then, solving recursively, in a fashion analogous to that used to derive eq. (6), the equilibrium rental payment for an n-payment operating lease which gives the lessee the right to purchase the asset for the price P_T at the maturity date of the contract can be determined as

$$L^{**}=\sum_{i=0}^{n-1}\lambda^i(1-\lambda)A_0\cdot N_i(h_i+\sigma\sqrt{i};\{\rho\})-L^{**}\sum_{i=1}^{n-1}R_f^{-i}\cdot N_i(h_i;\{\rho\})$$

$$+\lambda^n\cdot A_0\cdot N_n(h_n+\sigma\sqrt{n};\{\rho\})-P_T\cdot R_f^{-n}\cdot N_n(h_n;\{\rho\}), \tag{9}$$

where L^{**} is the equilibrium rental payment and other terms are as defined above.

The solution for the lease with a purchase option at maturity differs from that for the standard operating lease by the inclusion of the last two terms on the right-hand side of the equation and by the change in the boundary conditions reflected in L^{**}. The last two terms in eq. (9) are analogous to the Black–Scholes (1973) model for pricing options on non-dividend paying stocks. These terms differ only because the multivariate normal distribution has replaced the univariate distribution and because the underlying asset value is adjusted for expected economic depreciation. The addition of the last two terms means that the equilibrium rental payment for an n-payment lease which includes a purchase option will exceed those of a standard n-payment lease.

Implementation of the model for valuing the n-payment lease with a purchase option requires the same data as the valuation of a standard lease with the addition of the fixed purchase price P_T. A potential lessee would merely compute the equilibrium rental payment with eq. (9) and compare that with the proferred contractual lease payment. Alternatively, L^{**} could be shifted from the left-hand side of eq. (9) to the right-hand side, and the contractual lease payments, $\bar{L}$, could be substituted for L^{**} to estimate the net advantage to leasing.

3.4. Valuing the lease with option to purchase at any time

From the discussion in section 2.4, the lease with an option to purchase the asset at any time, where the purchase price declines after each lease payment by the amount of the payment, is equivalent to the lease with option to purchase only at maturity. Therefore, eq. (9) can be used to value any n-payment lease that contains an option to purchase the asset at any time at a price that declines through time by the amount of accumulated rental payments merely by substituting $\hat{P}_T$ for P_T, where $\hat{P}_T$ is the final purchase price under the lease with option to purchase at any time.

3.5. Valuing the lease with an asset purchase requirement

As discussed in section 2.5, under the open-end lease (or conditional sales lease) each of the options in a standard operating lease with an asset purchase option will be exercised with certainty. With the inclusion of this feature, eq. (9) reduces to

$$L^{**} = (1-\lambda^n)A_0 - L^{**}\sum_{i=1}^{n-1} R_f^{-i} + \lambda^n A_0 - P_T \cdot R_f^{-n}. \tag{10}$$

The first term on the right-hand side of (10) is the present value of the service flows generated by the asset from the initiation of the lease until time T and the third term is the present value of the service flows from time T to $+\infty$. The sum of those two terms is merely the current price of the asset so that eq. (10) can be rewritten as

$$L^{**} = A_0 - \sum_{i=1}^{n-1} L^{**}/(1+r_f)^i - P_T/(1+r_f)^n, \tag{11}$$

and the net advantage to leasing can be computed as

$$NAL = A_0 - \sum_{i=0}^{n-1} \bar{L}/(1+r_f)^i - P_T/(1+r_f)^n. \tag{12}$$

Notice here that each of the terms is discounted at the risk-free rate and there is no adjustment for uncertainty.[8]

[8]Care must be taken in interpreting eq. (12). The rental payments on the 'open-end' lease will be lower than those on the standard operating lease. This does not mean that the open-end lease is a 'cheap' form of financing. Use of other assets to collateralize the lease will raise the cost of acquiring funds to finance the other assets. Continual reliance on this strategy represents a form of the 'beggar-thy-lender' policy discussed by Kim et al. (1978). In a frictionless market the overall value of the firm will not be increased by this financing policy.

3.6. Valuating the lease with non-cancellation period

From section 2.6 an operating lease that contains a non-cancellation period can be valued as a standard operating lease on which each of the lease payments during the non-cancellation period will be made with certainty. Thus, from eq. (6), the equilibrium rental under an operating lease that matures at date T which contains a non-cancellation period extending from the initiation of the lease until date $T-K$ is

$$L^{**} = (1-\lambda^{n-k})A_0 - L^{**}\sum_{i=1}^{k-1} R_f^{-i} + \sum_{i=n-k}^{n-1} \lambda^i (1-\lambda) A_0 \cdot N_{i-n+k}(h_{i-n+k} + \sigma\sqrt{i-n+k}; \{\rho\}) - L^{**}\sum_{i=n-k}^{n-1} R_f^{-i} \cdot N_{i-n+k}(h_{i-n+k}; \{\rho\}). \tag{13}$$

The first two terms on the right-hand side of eq. (13) represent the value of the equilibrium lease payment during the non-cancellation period. The third and fourth terms represent the value of the lease payment during the period in which the lease is a standard operating lease. Of course, the rental payment for this lease is the same as that for a non-cancellable lease that matures at $T-K$ which includes an option to extend the lease to date T.

When the lease is non-cancellable for the entire life of the contract, eq. (13) reduces to

$$L^{**} = (1-\lambda^n)A_0 - L^{**}\sum_{i=1}^{n-1} R_f^{-i}. \tag{14}$$

Noting that $\lambda^n A_0$ represents the present value of the residual value of the asset at the maturity date of the contract, call it S_0^n, we can write the net advantage to leasing for the non-cancellable financial lease as

$$NAL = A_0 - \sum_{i=0}^{n-1} \bar{L}/(1+r_f)^i - S_0^n. \tag{15}$$

The NAL of eq. (15) is equivalent to the net advantage to leasing as defined in much of the extant leasing literature in the absence of taxes and default risk.[9]

[9]We have assumed that the firm pledges sufficient collateral to make the non-cancellable financial lease default-free. That may not be precisely correct. Even if the firm pledges its full faith and credit to support the lease, the firm itself may go bankrupt. Thus, there is still some probability (ex ante) that the lease payments will not be made. In that case the financial lease could still be valued as a compound option, but the boundary conditions would be altered to

In similar fashion, eq. (9) can be modified to value a non-cancellable financial lease which contains a purchase option with exercise price P_T. By incorporating the provision that each of the lease payments will be made with certainty, eq. (9) reduces to

$$L^{**}=(1-\lambda^n)A_0-L^{**}\sum_{i=1}^{n}R_f^{-i}+\lambda^n\cdot A_0\cdot N_1(h_1+\sigma\sqrt{n})-P_T\cdot R_f^{-n}\cdot N_1(h_1), \tag{16}$$

with net advantage to leasing of

$$NAL=(1-\lambda^n)A_0-\bar{L}\sum_{i=0}^{n-1}R_f^{-i}+\lambda^n\cdot A_0\cdot N_1(h_1+\sigma\sqrt{n})-P_T\cdot R_f^{-n}\cdot N_1(h_1). \tag{17}$$

Application of this particular model is relatively simple. The only option to be evaluated is the final purchase option which contains only a univariate normal distribution.

Additional variations on the various lease contracts that we have discussed can easily be imagined, and indeed, have appeared in actual lease contracts. Rather than present further formal solutions for these contracts, in the following section we examine the partial derivatives of the various terms included in the model and consider some numerical examples that demonstrate the way in which the models can be implemented. These examples also illustrate the impact of various parameters in the model on the amount of the equilibrium lease payments.

4. Illustrating and implementing the model

The partial derivatives of the equilibrium lease payment with respect to the various parameters of the model for valuing the standard operating lease are

$$\frac{\partial L^*}{\partial A},\ \frac{\partial L^*}{\partial \mathrm{E}(\tilde{d})},\ \frac{\partial L^*}{\partial R_f},\ \frac{\partial L^*}{\partial \sigma^2}>0 \quad \text{and} \quad \frac{\partial L^*}{\partial \sigma_{ty}}<0.$$

reflect the value of the firm's other assets. In principal the problem could be solved, but the solution would be messy. It would include the expected rate of depreciation (or appreciation) of the firm's other assets, the variance rate of the other assets, the covariance between the other assets and the leased asset, and the covariance between the firm's other assets and the market return [for example, see Schwarz (1981)]. If the firm had issued other financial claims with prior claims on some of the firm's assets, the solution would have to reflect the covenants of those claims as well [for example, see Black and Cox (1976)]. Alternatively, if the corporation is a single-asset firm, the financial lease would be valued as a standard operating lease in which the boundary conditions depend only upon the use value of the leased asset (assuming that all net cash flows are paid as dividends). We have finessed the issue by assuming that financial leases are, in fact, non-cancellable. The extant leasing literature finesses the issue by suggesting that the lease payments in eq. (16) be discounted at the firm's current borrowing rate. Of course, if the financial lease is default-free, both approaches give the same result.

Interpretation of the partials is reasonably straightforward. Specifically, the larger the initial value of the asset, the larger will be the lessor's capital investment and the larger will be the lease payment required to induce the lessor to invest in the asset. Similarly, the higher the expected rate of economic depreciation, the lower will be the expected residual value of the asset at each point in time in the future. Consequently, the lessor will demand a higher lease payment per period to compensate for the increased expected reduction in asset value.

An increase in the risk-free rate of interest means that lessors will demand larger lease payments to compensate for the increased opportunity cost of capital invested.

The variance rate measures the volatility of the asset's market value. At each decision point the lessee's decision to make the lease payment is contingent upon the asset's market value. As the asset's volatility increases, the probability that the asset's market value will be below the critical value increases. To compensate for the increased probability of cancellation, lessors will demand larger lease payments.

Finally, the covariance between the logarithm of one minus the rate of economic depreciation and the 'market factor' is a measure of the 'non-diversifiable' risk of the capital invested in the asset. The partial derivative of the equilibrium rental payment with respect to the covariance term is negative. The leased asset is normally a wasting resource so that a negative covariance between the rate of depreciation and the market factor will increase the lease payment because the certainty equivalent of the residual value is smaller. This occurs, in the words of Miller and Upton (1976, fn. 10), if 'necessity is indeed the mother of invention and the pace of technological improvement steps up as the economy falls off'. In other words, assets with negative covariance terms would tend to hold their value (or depreciate at lower rates as the economy improves). Conversely, a positive covariance term will decrease the lease payment. Miller and Upton argue this can result 'if it is boom conditions in the economy that stimulate technological progress'.[10]

The partial derivatives indicate the qualitative impact of changes in the parameters of the model on the equilibrium lease payment and (by implication) on the value (or *NAL*) of a lease contract. Some numerical examples will indicate the quantitative impact of changes in the parameters

[10]In two empirical studies of the leasing market Sorenson and Johnson (1977) and Crawford et al. (1981) regressed the estimated yields on the leases against the initial purchase price or 'cost' of the leased asset, the dollar value of the 'collateral' pledged to support the lease, the length of the time interval covered by the lease (in years), and the prepayment or initial downpayment required on the lease. The signs of the estimated coefficients yielded by the regressions are consistent with the partial derivatives of the derived lease valuation model. A useful future empirical study would include the expected rate of economic depreciation on the asset, the covariance between the asset's value and the market return, and the variance rate of the asset's value as additional explanatory variables.

of the model on equilibrium lease payments and the NAL.[11] For our basic example we will (arbitrarily) assume that: (1) the asset's initial market value is \$1,000; (2) its expected rate of economic depreciation is 15% per year; (3) the covariance between the asset's rate of economic depreciation and the market factor is zero; (4) the variance of the rate of change in asset's value is 15% per year; (5) the risk-free rate of interest is 10% per year; and (6) the lease calls for annual rental payments.[12] To compute the NAL it is necessary to specify a contractual lease payment. For that purpose we will assume a contractual lease payment of \$230 per year (i.e., $\bar{L}=\$230$).

Once we have computed the equilibrium lease payments (L^*) we will compute the 'yield-to-maturity' or internal rate of return on the lease by solving for Y in the following equation:

$$0=A_0-\sum_{t=0}^{T-1} L^*/(1+Y)^t-\bar{S}_T/(1+Y)^T, \tag{18}$$

where L^* is the computed equilibrium base payment, $\bar{S}_T$ is the expected residual value of the asset at the maturity date of the lease, and T is the number of annual rental payments under the lease. $\bar{S}_T$ is computed as \$1,000 $(1-d)^T$, where $d=0.15$ in the basic example. We shall compute the yield on the lease because the internal rate of return has been suggested by several authors as a means for determining the acceptability of alternative lease contracts. The trick, of course, is identifying the appropriate benchmark rate for comparison with the yield on the lease. The yields generated with the valuation model could serve as such a benchmark.

Results for the basic example are reported in panel A of table 1. Numerical solutions were generated for leases with maturities of 1 through 5 years. Column 2 of the table shows the equilibrium lease payment; column 3 contains the internal rate of return computed with the equilibrium lease payment; and column 4 gives the NAL computed with the arbitrarily chosen contractual lease payment of \$230 per year.

The equilibrium lease payment is less than \$230 for a 1-year lease and greater than \$230 for leases with maturities of 2 through 5 years. Consequently, the NAL is negative for a 1-year lease and positive for leases

[11]Geske (1977) shows the way in which the cumulative multivariate normal distribution function can be factored using an integral reduction process developed by Curnow and Durnet (1962) to solve for the value of the lease payments. However, in this paper we use a computer program developed by Milton (1972) to evaluate the multivariate normal integral.

[12]The parameter values were chosen arbitrarily, but not randomly. In particular, Fama (1976) reports the standard deviations in monthly returns for 30 randomly selected companies listed on the New York Stock Exchange. When those estimates are converted to annualized variances the range is 0.94% to 31.88%. The variance of the rate of change in the asset is approximately the mid-point of that range. The expected rate of economic depreciation that is used implies that the expected salvage value of the asset will be slightly less than one-half its original value at the end of five years and will be approximately 20% of the original value at the end of 10 years.

Table 1
Lease payment, yield and *NAL* for standard operating lease contracts with various parameter values.

	Term to maturity (years)	Equilibrium lease payment (dollars/year)	Yield-to-maturity (percent/year)	*NAL* (dollars)
(A)[a]	1	$227.27	10.0%	$-2.73
	2	240.64	14.7	12.97
	3	248.36	18.8	24.80
	4	253.29	22.4	33.45
	5	256.59	25.5	39.83
(B)[b]	1	$227.27	10.0%	$-2.73
	2	232.22	13.3	2.59
	3	234.44	16.3	5.52
	4	235.57	19.0	7.14
	5	236.17	21.4	8.04
(C)[c]	1	$227.27	10.0%	$-2.73
	2	246.88	15.7	20.74
	3	259.07	20.8	39.88
	4	267.36	25.2	54.86
	5	273.20	29.0	66.43
(D)[d]	1	$136.36	10.0%	$-93.64
	2	148.37	12.0	-91.81
	3	156.96	13.7	-89.34
	4	163.68	15.1	-86.58
	5	169.09	16.4	-83.75
(E)[e]	1	$318.18	10.0%	$88.18
	2	329.28	19.4	124.96
	3	333.96	28.0	142.76
	4	336.16	35.1	151.68
	5	337.23	40.5	156.09

[a]Basic example: Solution with parameter values of asset price = $1,000; risk-free rate = 10% per year; expected rate of economic depreciation = 15% per year; variance rate of change in asset value = 15% per year; covariance rate = 0; contractual lease payment = $230 per year.

[b]Solutions with variance rate = 5% per year; other parameter values same as panel A.

[c]Solutions with variance rate = 25% per year; other parameter values same as panel A.

[d]Solutions with expected rate of depreciation = 5% per year; other parameter values same as panel A.

[e]Solutions with expected rate of depreciation = 25% per year; other parameter values same as panel A.

with maturities of 2 through 5 years. Perhaps the most interesting result is that the yields exceed the risk-free rate by a wide margin for all maturities. For the 5-year lease the yield is 25.5% (vs. a risk-free rate of 10%). The apparent implication is that, because of the nature of the contract, yields on

operating leases, as traditionally measured, exceed those on high-grade bonds by substantial margins even in equilibrium and in a perfect market.

Of course, the results in panel A depend upon the assumed values of the parameters. To give some indication of the sensitivity of the results to changes in the values of the parameters, the variance rate and the expected rate of economic depreciation were varied. The results are presented in panels B through E of table 1. Panels B and C report the results when the variance of the rate of change in asset value is 5% and 25% per year, respectively. Panels D and E present the results when the expected rate of depreciation is 5% and 25%. In each case, the values of the other parameters are the same as those in the basic example, panel A.

When the variance of the rate of change in asset value is 5% per year, the equilibrium lease payment is reduced, but it is still greater than \$230 per year for all but the 1-year lease. Consequently, the *NAL* is positive for all but the 1-year lease as well. Furthermore, the yields are still significantly above the risk-free rate. For the 5-year lease the yield is 21.4%. Similar results are generated when the variance rate is 25% except that the lease payments, *NAL*s and yields are increased for leases of every maturity.

The results in panels D and E emphasize the importance of the expected rate of economic depreciation on equilibrium rental payments. In panel D, when the expected rate of depreciation is 5%, the equilibrium lease payment is less than the contractual lease payment for leases of all maturities and the *NAL* is negative throughout. Even here, however, when the equilibrium lease payments are lower than in any of the other simulations the yields are substantially above the risk-free rate — for the 5-year lease the yield is 16.4%. Increasing the rate of economic depreciation to 25% from 5% more than doubles the equilibrium lease payment and causes the *NAL* to be positive for all leases. The increase in the depreciation rate also has a dramatic impact on the yield — for the 5-year lease the yield is 40.5%. When the rate of asset depreciation is high and when the lessee may cancel the lease, the lessor will require very high lease payments early in the life of the lease to compensate for the loss in asset value.

Results were also generated for leases which contain an option to purchase the leased asset at the maturity date of the lease. A selection of these results is presented in table 2. Panel A of the table contains results for 4-payment leases with fixed purchase prices of \$200, \$300, \$400, \$500, and \$600; all other parameter values are the same as in the basic example. The parameter values used to generate the results in panel B are the same as those in panel A except that the variance rate of the asset is assumed to be 25% per year. Finally, panel C presents solutions for 5-payment leases with the same parameter values as those in the basic example, panel A. To give some idea of the relationship between the exercise price of the option and the expected residual value of the asset with an assumed expected depreciation rate of

Table 2
Lease payment, yield and *NAL* for operating lease contracts with option to purchase at maturity.

	Term to maturity (years)	Purchase price at maturity (dollars)	Equilibrium lease payment (dollars/year)	Yield-to-maturity (percent/year)	*NAL* (dollars)
(A)[a]	4	$200	$309.86	34.4%	$159.75
	4	300	299.92	32.1	133.96
	4	400	292.15	30.4	114.62
	4	500	286.29	29.2	98.36
	4	600	281.04	28.0	87.01
(B)[b]	4	$200	$336.54	40.7%	$202.38
	4	300	327.18	38.4	179.92
	4	400	320.85	36.9	162.53
	4	500	314.87	35.5	148.61
	4	600	310.18	34.4	137.39
(C)[c]	5	$200	$293.32	33.3%	$119.98
	5	300	288.05	32.2	107.38
	5	400	284.57	31.4	97.24
	5	500	281.48	30.7	88.86
	5	600	278.49	30.1	82.02

[a]Solutions with parameter values same as for the basic example (panel A of table 1).
[b]Solutions with variance rate = 25% per year; other parameter values as basic example.
[c]Solutions with parameter values same as basic example.

15% per year, the residual value of the asset is expected to be $522 after 4 years and it is expected to be $444 after 5 years.

The results in table 2 show that the purchase option increases the equilibrium lease payment, the yield of the lease, and the *NAL*. Indeed, for all combinations of parameters shown the equilibrium lease payment exceeds the assumed contractual lease payment of $230 so that the *NAL* is positive throughout. Yields on the leases, which range from 28.0% to 40.7%, exceed the risk-free rate by significant margins. Of course (as the table indicates), increases in the fixed exercise price of the purchase option decrease the equilibrium lease payments, yields, and *NAL*s whereas increases in the variance rate of the asset increase the lease payments, yields, and *NAL*s.

There is one significant element in these results. For the parameter values used here, an increase in the term-to-maturity of the lease gives rise to a reduction in the equilibrium lease payment. In the absence of the purchase option, an increase in the term-to-maturity increases the lease payment. However, when the option exercise price is fixed (and the asset is depreciating), the purchase option becomes less valuable as the term over which the asset is depreciating is increased. This reduces the lease payment. In this example, the effect of the loss in the value of the purchase option is dominant and the net effect is a decline in the equilibrium lease payment as

the term-to-maturity is increased. That is, the partial derivative of the equilibrium lease payment with respect to term-to-maturity on an operating lease with purchase option may be positive or negative depending upon the values of the other parameters.

The partial derivatives of the equilibrium rental payment on the non-cancellable financial lease are also the same as those for the standard operating lease with the exception of the partial with respect to term-to-maturity. With a financial lease, the equilibrium lease payments decline as the term-to-maturity increases. The reason for this is as follows: The contractual lease payments are specified to be the same each period, but as the term of the lease is increased the expected asset value declines. Because the use value of the asset is a constant fraction of the asset value, the amount of each market-determined lease payment will be less than the previous one. Thus, each additional lease payment will reduce the size of the average lease payment. Because the contractually specified lease payments are equal, adding one more (lower) lease payment reduces all the lease payments.[13]

To illustrate this point, some numerical results for non-cancellable financial leases are presented in table 3. Panel A contains results with all parameter values the same as in the basic example; in panel B the expected rate of economic depreciation is 25% per year and other parameter values are the same as panel A. In each case the equilibrium lease payment is less than the lease payment under the corresponding standard operating lease and, as we discussed, the equilibrium lease payments decline as the maturity of the lease is lengthened.

Additionally, for each lease the yield-to-maturity is equal to the risk-free rate of 10% per year. This occurs because the lease payments are default-free *and* because we have assumed that the covariance between the rate of change in the asset's value and the market index is zero.[14] Thus, there is no 'non-diversifiable' risk associated with the asset's residual value. We set the covariance term to zero to illustrate that the relatively high yields (and equilibrium lease payments) under the standard operating leases are not due to non-diversifiable market risk, but rather to the value (or risk) associated with the compound option imbedded in the standard operating lease. These examples emphasize the point.

Panels C and D of table 3 illustrate the impact of residual value risk on the terms of the financial lease. In panels C and D the values of all parameters are the same as in panel A except that the covariance between the rate of change in the asset's value and the market index is set at -0.5% and -4.0%, respectively. As discussed above, the partial derivative of the lease payment with respect to the covariance term is negative. Thus, the

[13]We are grateful to the referee for suggesting this point to us.

[14]More precisely the covariance between the logarithm of one minus the rate of economic depreciation and the market factor is set to zero.

Table 3
Lease payment, yield and *NAL* for non-cancellable financial leasing contracts.

	Term to maturity (years)	Equilibrium lease payment (dollars/year)	Yield-to-maturity (percent/year)	*NAL* (dollars)
(A)[a]	1	$227.27	10.0%	$-2.73
	2	211.04	10.0	-36.20
	3	196.89	10.0	-90.57
	4	184.54	10.0	-158.51
	5	173.75	10.0	-234.58
(B)[b]	1	$318.18	10.0%	$88.18
	2	280.30	10.0	96.03
	3	249.69	10.0	53.87
	4	224.81	10.0	-18.09
	5	204.48	10.0	-106.42
(C)[c]	1	$231.13	10.6%	$1.13
	2	214.15	10.5	-30.26
	3	199.40	10.4	-80.71
	4	186.56	10.3	-151.45
	5	175.38	10.3	-227.77
(D)[d]	1	$257.57	14.5%	$20.57
	2	235.09	13.7	-9.91
	3	215.96	13.1	-38.40
	4	199.66	12.6	-105.80
	5	185.72	12.1	-184.63

[a]Solutions with parameter values same as for the basic example (panel A of table 1).
[b]Solutions with expected depreciation rate = 25%; other parameter values same as basic example.
[c]Solutions with $\sigma_{ly} = -0.5\%$; other parameter values same as basic example.
[d]Solutions with $\sigma_{ly} = -4.0\%$; other parameter values same as basic example.

effect of a negative covariance term is to increase equilibrium rental payments, to decrease *NAL*s and to increase yields. The smaller (negative) covariance term in panel D results in yields that are 2 to 4 percent above the riskless rate. The closer the covariance term is to the zero, the closer the yield is to the riskless rate. Likewise, and for similar reasoning, the yield-to-maturity declines as the maturity of the contract is increased. Asymptotically, the yield approaches the risk-free rate as maturity is increased.[15]

[15]If the covariance term is positive, the equilibrium lease payments will be less than those when the covariance term is zero and the yields will be less than the risk-free rate. The yields will increase with maturity and asymptotically approach the riskless rate.

5. Summary and conclusion

The problem of valuing asset leasing contracts has vexed financial managers and academic investigators for some time. The problem is especially interesting to researchers in the area of managerial finance because it combines some aspects of a capital budgeting decision with some aspects of a financing decision.

In this paper we develop a model for valuing leases in a multi-period framework. To do so, we first observe that a 'standard' operating leasing contract is a compound option in which each lease payment but the last purchases the use of the asset over some time interval and it also purchase options to purchase the use of the asset over future time intervals. The exercise prices of the future options are the subsequent periodic rental payments. We then invoke Rubinstein's (1976) method for valuing risky income streams and Geske's (1977) method for valuing compound options to derive a model for valuing a 'standard' operating lease.

We also discuss the relationship between the standard operating lease and a variety of other types of leasing contracts and show the way in which these leases can be valued directly with our model or with some slight modifications to it. In particular, we show that the model (or a slight variation of it) can be used to value: (1) leases that give the lessee an option to purchase the leased asset at a specified price at the maturity date of the lease; (2) leases that allow the lessee to purchase the asset at any time during the life of the lease at a price that declines through time by the amount of the accumulated lease payments; (3) leases that permit the lessee to extend the life of the lease — the renewable lease; (4) leases that require the lessee to purchase the underlying asset for a fixed price at the maturity date of the lease — the open-end or conditional sales lease; and (5) leases that contain a non-cancellation period.

Implementation of the model requires knowledge of the asset's initial market price and the risk-free rate of interest along with an estimate of the asset's expected rate of economic depreciation, the covariance between the logarithm of one minus the asset's rate of economic depreciation and a market factor, and the variance rate of the asset's market value through time.

Finally, we demonstrate the way in which the model can be used by lessors to set equilibrium lease payments and by lessees when evaluating alternative leasing arrangements or when making lease vs. borrow-and-buy decisions. This demonstration involves an evaluation of some hypothetical standard operating contracts, some lease contracts which contain a purchase option, and some non-cancellable financial leases. These examples help to illustrate the differences between two types of risks inherent in lease contracts — cancellation risk and residual value risk — and the impact of each on equilibrium rental rates.

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