Rational Prepayments and the Valuation of Collateralized Mortgage Obligations
Author(s): John J. McConnell and Manoj Singh
Source: *The Journal of Finance*, Vol. 49, No. 3, Papers and Proceedings Fifty-Fourth Annual Meeting of the American Finance Association, Boston, Massachusetts, January 3-5, 1994 (Jul., 1994), pp. 891-921
Published by: Wiley for the American Finance Association
Stable URL: http://www.jstor.org/stable/2329210
Accessed: 10-02-2016 19:41 UTC

THE JOURNAL OF FINANCE • VOL. XLIX, NO. 3 • JULY 1994

Rational Prepayments and the Valuation of Collateralized Mortgage Obligations

JOHN J. MCCONNELL and MANOJ SINGH*

ABSTRACT

This article presents a procedure for evaluating collateralized mortgage obligation (CMO) tranches. The solution procedure is in the spirit of a dynamic programming problem in which an individual mortgagor's decision to prepay is the feedback control variable—the mortgagor seeks to minimize the value of the mortgage subject to refinancing costs. We employ a two-step procedure to solve this dynamic programming problem. The first step uses an implicit finite difference backward solution procedure to determine the "optimal" prepayment boundary for a class of mortgagors, each of whom confronts the same proportional refinancing cost. This step is repeated for several different classes of mortgagors that differ in the level of refinancing costs that they confront. The outcome of this first step is a series of prepayment boundaries—one set of boundaries for each level of refinancing costs (i.e., one set of boundaries for each refinancing cost category of mortgagors). In the second step, the prepayment boundaries determined in the first step are used in conjunction with Monte Carlo simulation to value the CMO tranches. The essence of the second step is that when the simulated interest rate hits the boundary for a particular class, it triggers a prepayment scenario for that class of mortgagors. We conduct extensive sensitivity analysis to determine the robustness of this approach (and our solution procedure) to alternative single-factor models of the term structure of interest rates and to alternative specifications of the distribution of refinancing cost levels confronted by mortgagors. The sensitivity analysis indicates that CMO tranche valuation is not particularly sensitive to alternative models of the term structure so long as the models are consistent with the current yield curve, but, even when alternative specifications of the refinancing cost categories generate nearly identical values for the collateral underlying the CMO (i.e., the generic mortgage-backed securities), the resulting tranche values can differ widely between the two specifications. The results point out the importance of accurate estimation of the distribution of refinancing costs when the rational valuation model is used for the analysis of CMO tranches.

AS OF SEPTEMBER 1993, the face amount of outstanding securitized mortgage instruments exceeded $1.2 trillion. Of this total, the vast majority had been transformed from generic mortgage-backed securities (MBS) into collateralized mortgage obligations (CMO) in which the cash flows (i.e., the principal and interest payments) from the underlying mortgages are allocated among various tranches or classes according to a preestablished, although sometimes complex, hierarchy. Within a CMO, the various tranches differ from

*McConnell is from Purdue University. Singh is from Boston College. This article has benefitted from many discussions with Stephen Ross and from comments by Scott Richard.

each other according to the priority of cash flows received and according to the degree to which the tranches have claims against either principal, interest, or both. Thus, *within* a CMO, even though two tranches are supported by the same collateral and provide the same coupon rate of interest, the prices (and "yields") of these two tranches may differ because the timing and amounts of principal payments are uncertain.

Similarly, *across* CMOs, two otherwise apparently identical tranches may differ in price (and "yield") because the coupon or remaining term to maturity of the underlying collateral differs between the two structures. For example, given two identically structured CMOs, the A tranches of the two may differ in price (and "yield") because the coupon rates or the remaining terms-to-maturity of the underlying collateral differ between the two. Thus, even though the trading volume of CMO tranches is substantial, most trades involve analysis of the specific tranche in question. That is, CMO tranches typically do not trade as generic instruments. For that reason, significant resources have been devoted to the development of "mortgage analytics."

Two components are fundamental to the analysis of mortgage-related instruments—specification of the dynamics of the term structure of interest rates and characterization of the call option exercise policy followed by mortgagors. In this regard, development of mortgage valuation models has evolved along two related, but, still separate, paths. They are related in the way in which they incorporate the dynamics of the term structure of interest rates—both rely upon the assumption of an arbitrage-free economy and, within this no-arbitrage framework, both rely upon one of several well-known models of the term structure of interest rates. These models include, among others, Brennan and Schwartz (1982), Cox, Ingersoll, and Ross (1985), Heath, Jarrow, and Morton (1992), Ho and Lee (1986), Richard (1978), and Vasicek (1977).

The two evolutionary paths differ in the way in which they characterize the call option exercise policy followed by mortgagors. The first path assumes that mortgagors follow an "optimal" call policy constrained by some level of transactions costs. This set of research includes models developed by Dunn and McConnell (1981a, 1981b), Dunn and Spatt (1986), Johnston and Van Drunen (1988), Kau, Keenan, Muller, and Epperson (1992), and Stanton (1993a, 1993b).[1] In these models, mortgagors exercise their call option whenever the value of the mortgage, if left uncalled, would exceed the remaining principal balance of the loan plus the transactions costs associated with refinancing it. Stanton (1993a) refers to this class of models as "rational" valuation models.[2] The virtue of this class of models is that they are derived under the fundamental premise that individual mortgagors follow a con-

[1] Dunn and McConnell (1981a, b) effectively assume that transaction costs are zero.

[2] These models are known by other names as well. For example, Merrill-Lynch has developed a version of the Johnston and Van Drunen (1988) model to which they refer as the "refinancing threshold pricing" (RTP) model. That is, the benefit of refinancing must exceed a threshold level of refinancing costs before the mortgagor will repay his loan.

strained utility-maximizing call policy. As a consequence, Stanton argues that this approach is robust to structural changes in the economic environment. The deficiency of this approach is that it does not immediately lend itself to the valuation of CMO tranches. In particular, rational prepayment models rely upon a finite difference backward solution procedure that begins with the maturity date of the mortgage and proceeds backwards in time under the assumption that, at each point in time, mortgagors follow the optimal call policy based upon future cash flows. This solution procedure does not lend itself to analysis of CMOs because the hierarchical structure of payments to the various tranches requires knowledge of prior mortgage prepayments. That is, the backward finite difference solution procedure does not readily accommodate the "memory" required to determine the allocation of cash flows among the CMO tranches.

The alternative evolutionary path relies purely upon statistical estimation to characterize mortgagor call policy. Under this approach, a "prepayment" model is estimated as a function of interest rates, other macroeconomics variables, and certain pool-specific variables. The virtue of the purely statistical approach is that it can be easily adapted to the analysis of CMOs. In this type of analysis, the empirically estimated prepayment model is used with a "forward-looking" Monte Carlo simulation to value the CMO tranches. Because the Monte Carlo procedure begins at time zero and proceeds forward in time, it can keep track of prepayments and, therefore, the amount of principal that has been paid to each tranche at each point in time. In comparison with the rational prepayment models, however, the purely statistically based models may be less robust to changes in the economic environment.

Thus, application of the rational prepayment approach to CMOs would be a desirable enhancement of current mortgage analytics. It is to this task that we turn in this article. In particular, we present a solution procedure through which the rational prepayment approach can be used to analyze CMOs. The solution procedure is in the spirit of a dynamic programming problem in which an individual mortgagor's decision to prepay is the feedback control variable—the mortgagor seeks to minimize the value of the mortgage subject to refinancing costs. We employ a two-step procedure to solve this dynamic programming problem. The first step uses an implicit finite difference backward solution procedure to determine the "optimal" prepayment boundary for a class of mortgagors, each of whom confronts the same proportional refinancing cost. This step is repeated for several different classes of mortgagors that differ in the level of refinancing costs that they confront. The outcome of this first step is a series of prepayment boundaries—one set of boundaries for each level of refinancing costs (i.e., one set of boundaries for each financing cost category of mortgagors). In the second step, the prepayment boundaries determined in the first step are used in conjunction with Monte Carlo simulation to value the CMO tranches. The essence of the second step is that when the simulated interest rate hits the boundary for a particular class, it triggers a prepayment scenario for that class of mortgagors.

We should note that we are not claiming that the rational prepayment approach is necessarily superior to the purely empirically based approach. After all, even the rational prepayment procedure does require statistical estimation of the parameters of the distribution of refinancing costs. The conclusion as to whether one valuation technique dominates the other requires further experimentation and a comparison of the models over time. Stanton (1993a, 1993b) does, however, present evidence with a limited set of data for generic MBSs that the rational valuation approach is as accurate as at least one empirically based model and that the approach can successfully accommodate heterogeneity across mortgage pools. In short, the initial evidence appears promising.

We further the understanding of the rational valuation method for CMOs by conducting extensive sensitivity analysis to determine the robustness of this approach (and our solution procedure) to alternative single-factor models of the term structure of interest rates and to alternative specifications of the distribution of refinancing cost levels confronted by mortgagors. The sensitivity analysis indicates that CMO tranche valuation is not particularly sensitive to alternative models of the term structure so long as the models are consistent with the current yield curve, but, even when alternative specifications of the refinancing cost categories yield nearly identical values for the collateral underlying the CMO (i.e., the generic MBSs), the resulting tranche values can differ widely between the two specifications. The results point out the importance of accurate estimation of the distribution of refinancing costs when the rational valuation model is used for the analysis of CMO tranches.

The following section briefly describes CMO structures. Section II outlines the rational prepayment valuation procedure in greater detail. Section III presents our two-step procedure for valuation of CMOs using the rational prepayment method and conducts sensitivity analysis with respect to alternative specifications of the term structure and with respect to alternative specifications of the costs of refinancing. Section IV concludes.

I. The Structure of Collateralized Mortgage Obligations

CMOs have been created that have as many as 70 different tranches. Essentially, the CMO structure is a mechanism for reallocating mortgage call or prepayment risk among classes of investors. Typically, a CMO is collateralized by pools of mortgages with coupon rates and maturities that lie within a narrow range. Each period a servicing fee is subtracted from the principal and interest cash flows that are then passed through to investors. The CMO structure assigns the principal and interest cash flows from the underlying collateral to the various tranches.

To illustrate, consider a simple "sequential pay" CMO with five classes: an A tranche, a B tranche, a C tranche, an interest accrual or "Z-bond," and an interest only (IO) strip. The A, B, and C tranches are assigned a fixed rate of interest and a face value that may be any feasible rate and any feasible face value. That is, the coupon rates and face values must be feasible given the cash flows available from the underlying collateral. Because this CMO has an interest-only tranche, the coupon rates on the A, B, and C tranches must be less than the rate on the underlying collateral (the "excess" interest is passed to the IO strip). Under the sequential pay structure, the promised fixed rate of interest along with all principal payments (including any prepayments) from the collateral are paid to the A tranche until that tranche is fully retired. During this period, the B and C tranches receive only interest. Once the A tranche is retired all principal payments along with fixed rate of interest are paid to the B tranche until it is fully retired, after which time, all principal payments are then passed through to the C tranche.

The interest accrual or Z-bond also has a stated principal balance and a fixed rate of interest. However, the Z-bond does not receive any cash flows until the senior tranches are fully retired. Rather, interest payments that would have been paid to the Z-bond are paid to the senior tranches, in order of priority, until their principal balances are retired. Concurrently, the principal balance of the Z-bond is increased by the amount of the foregone interest payment. That is, the unpaid interest accrues to the Z-bond. Once the senior tranches are retired, principal payments from the underlying collateral are paid to the Z-bond along with the stated interest rate on the unpaid principal balance. Thus, the Z-bond is much like a zero coupon bond (hence, its name) except that the maturity is uncertain.

As its name implies, the IO tranche receives interest payments only, and these come about only so long as the coupon rate on the priority tranches is less than the rate on the underlying collateral. The "extra" interest is passed through to the IO tranche. The IO tranche continues to receive cash flows only so long as some principal remains outstanding from the underlying collateral. Once the collateral is fully retired, cash flows to the IO tranche cease.

As even this simple example illustrates, tracking and allocating the cash flows from the generic MBSs to the various tranches is critical to the analysis of CMOs. Constructing a valuation model that will do that in a rational prepayment framework is the goal of this article.

II. Rational Prepayments and Mortgage Valuation

A. The Foundation

Development of the rational valuation approach to mortgage valuation can be traced, at least, to Dunn and McConnell (1981a, 1981b). They assume that markets are frictionless and that mortgagors exercise their call option as soon as the value of the mortgage, if left uncalled, would exceed the face value of the loan. In their framework, and, indeed, in all of the subsequent rational mortgage valuation models, the value of a generic MBS is merely the sum of the values of individual mortgages that support it less the value of the fees associated with servicing and insuring the mortgages. An outcome of this model is that MBSs never sell at prices above par.

Given the empirical observation that MBSs do trade at prices above par, Dunn and Spatt (1986) and Johnston and Van Drunen (1988) incorporate the transaction costs associated with mortgage refinancing into the rational valuation framework. In their models, mortgagors do not prepay until the savings in interest cost associated with prepayment are sufficiently large to cover the cost of refinancing their loans. As a consequence, MBSs can trade at prices above par. Additionally, these models allow for differences in the cost of refinancing among different groups of mortgagors. The differential in transactions costs across mortgagors means that some mortgagors prepay sooner than others when interest rates decline—the lower the cost of refinancing for a class of mortgagors, the more sensitive that class is to a decline in the interest rate. This heterogeneity in costs and, therefore, speed of prepayment, gives rise to a pattern of prepayments that captures the "burnout" effect in mortgage prepayments.[3] Unfortunately, unreasonably high levels of refinancing costs are required to generate MBS prices as high as observed prices. To explain this phenomenon, Dunn and Spatt (1986) hypothesize that mortgagors do not respond immediately when interest rates decline. This lag in the prepayment process means that not all mortgagors prepay when the interest rate declines to the "optimal" refinancing level. The effect of this delay is that MBS prices are higher than they otherwise would be for any given level of refinancing costs and any level of interest rates.

In addition to the "optimal" prepayments that occur when interest rates decline, each of these rational models allows for a "background" level of prepayments that are unrelated to interest rate movements. Dunn and McConnell (1981a, 1981b) attribute such prepayments to exogenous phenomena such as relocations, divorces, deaths, and so on.[4]

Stanton (1993a, 1993b) operationalizes the various aspects of prepayment behavior. To account for "delayed" prepayments when rates decline, he posits that mortgagors evaluate their prepayment options only at discrete intervals rather than continuously. To operationalize this aspect of prepayment behavior, he introduces a probability function such that only a fraction of mortgagors prepay when the interest rate declines to the otherwise "optimal" refinancing level. To operationalize the background level of prepayments that occurs even when the interest rate has not declined to the optimal refinancing level, he introduces a constant rate of prepayments (a hazard rate) that occurs regardless of the level of interest rates. He then specifies the distributions of refinancing costs, the prepayment delay process, and the background

[3]Once an MBS is outstanding, interest rates may decline and prepayments occur. Then interest rates may rise. Subsequently, when interest rates fall again, prepayments will be less than the first time that rates fell to the current level because the most interest-sensitive mortgagors have already "burned out" of the pool.

[4]Johnston and Van Drunen (1988) generate a level of prepayments in the absence of changes in interest rates by assuming that each "transactions cost class" of mortgagors prepays at a fixed time, even if rates have not fallen to the "threshold" level required for prepayment by the class.

hazard rate of prepayments and estimates the parameters of each process for a sample of generic MBSs. He then uses these to value generic MBSs and finds that the model produces results comparable to those generated with the Schwartz and Torous (1989) empirically based model for MBS valuation.

Needless to say, our adaptation of the rational valuation approach for analyzing CMOs owes much to the prior work. In particular, we employ the rational valuation approach with refinancing costs along with a finite difference solution technique in our first step to obtain refinancing boundaries for each class of homogeneous mortgagors (where a class is distinguished by the level of refinancing cost that it faces). At each point in time, the refinancing boundary is a critical interest rate such that if the loan is not called, the value of the cash flows to be paid by the mortgagor is greater than the remaining principal balance of the loan plus the refinancing costs—we call these interest-rate-related prepayments.

The second step encompasses several parts. First, we introduce a background level of prepayments that occurs regardless of the level of interest rates. Second, we incorporate the idea that mortgagors make refinancing decisions only at discrete intervals. For this reason, only a fraction of the class of mortgagors prepays when the interest rate hits the critical level determined in step one. Third, we develop an algorithm for allocating cash flows from the underlying MBS collateral among the various CMO tranches. Finally, a Monte Carlo procedure is used to generate paths of interest rates. At each point in time, for each simulated path, if the interest rate is greater than the critical level determined in step one, only "background" prepayments occur. If the interest rate falls below the critical level, prepayments are the sum of the background prepayments and interest-rate-related prepayments. The level of interest-rate-related prepayments is less than 100 percent because the delay in the prepayment process means that not all mortgagors prepay when the interest rate falls to the otherwise "optimal" refinancing rate.

B. *The Valuation Procedure*

As with other models for mortgage valuation, the procedure that we propose for analyzing CMO tranches can be applied with any arbitrage-free model of the term structure of interest rates. For purposes of presentation, we employ the extended version of the one-factor model of Cox, Ingersoll, and Ross (1985) as exposited by Hull and White (1990). In this model, the short-term interest rate, r, follows the process

$$dr = [\theta(t) + a(t)(b - r)]dt + \sigma(t)\sqrt{r}\, dz,$$

where dr represents an infinitesimal change in r over an infinitesimal time period, dt, and dz is a standard Wiener process. The term $\sigma(t)\sqrt{r}$ is the instantaneous standard deviation of changes in r and the term $\theta(t) + a(t)$ $(b - r)$ represents the time-dependent, mean-reverting drift of the short-term interest rate. In this model, the market price of interest rate risk is $\varphi(t)\sqrt{r}$ for a function of time $\varphi(t)$. The risk-adjusted interest rate process is

$$dr = [\phi(t) - \alpha(t)r]dt + \sigma(t)\sqrt{r}\,dz, \tag{1}$$

where $\phi(t) = a(t)b + \theta(t)$, $\alpha(t) = a(t) + \varphi(t)\sigma(t)$, $\phi(t)$ and $\alpha(t)$ are time-varying drift parameters, and $\sigma(t)$ is a time-varying volatility parameter.

Let i represent the ith refinancing cost category of mortgagors, each of whom confronts refinancing cost RF^i, where RF^i is expressed as a fraction of the remaining principal balance of the loan. Let c_s be the servicing and insurance fee, also expressed as a proportion of the remaining principal balance of the loan.[5] Let $V^i(r, t)$ represent the value of the cash flows paid by mortgagors (i.e., the value of the mortgagors' liabilities), and let $M^i(r, t)$ denote the value of the cash flows received by the holder of the securitized mortgages (i.e., the value of the MBSs). Because the investor does not receive the cash flows due to refinancing (i.e., the refinancing costs) or the servicing and insurance fee, the value of the cash flows paid by mortgagors always exceeds the value of the cash flows received by investors, i.e., $V^i(r, t) > M^i(r, t)$. Given the model of the term structure in equation (1) and invoking the fundamental valuation equation, the value of an MBS collateralized by a pool of mortgages from refinancing cost category i is governed by the system of equations

$$1/2r\sigma(t)^2 V^i_{rr} + (\phi(t) - \alpha(t)r)V^i_r + V^i_t + A^i - rV^i = 0, \tag{2a}$$

$$1/2r\sigma(t)^2 M^i_{rr} + (\phi(t) - \alpha(t)r)M^i_t + M^i_i + (A^i - c_s F^i(t)) - rM^i = 0 \tag{2b}$$

where A^i is the continuous time analog of the mortgage annuity payment comprised of the interest and principal payments, and $F^i(t)$ is the remaining principal of the loan at time t.

Under the assumption that mortgagors continuously evaluate their refinancing alternatives, the prepayment boundary condition for mortgagors from refinancing cost category i is

$$V^i(r, t) \leq (1 + RF^i)F^i(t). \tag{3}$$

Under the same assumption, the boundary condition for the value of the securitized mortgages (i.e., the MBS) is

$$M^i(r, t) = F^i(t), \quad \text{whenever} \quad V^i(r, t) = (1 + RF^i)F^i(t). \tag{4}$$

With these boundary conditions and equations (2a) and (2b), a finite difference solution procedure can be used to determine the value of a generic MBS collateralized by a pool of mortgages all of which carry the *same* coupon rate of interest, the same maturity, and all of which are from the same refinancing cost category i. Since equations (2a) and (2b) are second order, partial, differential equations, the solution procedure requires one terminal condition

[5]The servicing fee is retained by the loan originator for processing the paperwork associated with the loans and for collecting and passing through the loan payments.

and two natural boundary conditions. The terminal condition is that the value of the mortgagors' liability as well as the value of the pass-through MBS are equal to zero at maturity, or, $V^i(r, T) = M^i(r, T) = 0$. The first natural boundary condition is that the values go to zero as interest rates go to infinity, or, $V^i(\infty, t) = M^i(\infty, t) = 0$. The second boundary condition is the "high contact" boundary condition, which ensures a smooth slope for the values of $V^i(r, t)$ and $M^i(r, t)$ at the critical interest rate, r_c when equation (4) holds. This boundary condition is $V_r^i(r_c, t) = 0$, where the subscript, r, denotes the first derivative with respect to r. Analysis of a generic MBS supported by mortgages from several *different* refinancing cost categories is straightforward because the value of the MBS is merely a weighted average of the values of the securitized mortgages from the various refinancing cost categories. The weights are the remaining principal balances of the various refinancing cost categories relative to the total remaining principal balance of the underlying collateral.

It is here that our procedure for analyzing CMOs diverges from the rational prepayment models developed for the valuation of generic MBSs. Our model diverges because the sequential assignment of cash flows within a CMO structure requires knowledge of the history of mortgage prepayments—a capacity that is not technically feasible with the finite difference solution procedure.[6] The second step of our procedure does, however, rely upon the finite difference solution procedure and equation (2a) to determine a critical boundary of interest rates, $r_c^i(t)$, at which it is "optimal" for a group of mortgagors facing refinancing cost RF^i to prepay. That is, the mortgagor's decision to prepay is the feedback control variable in the second step of our solution procedure. To determine this critical boundary, let $r_c^i(t)$ be the interest rate at time, t, for refinancing cost category i obtained as the solution to the equation $V^i(r_c^i, t) = (1 + RF^i)F^i(t)$.

To extend the model to incorporate background prepayments and the delay in the prepayment process, we adopt the framework proposed by Stanton (1993a). As does he, we define two parameters associated with prepayments. The first is $\lambda(t)$, a time-dependent prepayment hazard rate that determines the level of background prepayments in the absence of interest-rate-related refinancing prepayments. The second parameter, ρ, captures the observed delay in refinancing activity attributable to the idea that mortgagors evaluate their refinancing alternatives only at discrete intervals. Over any time interval, the probability of prepayment depends on the level of interest rates and the time-dependent hazard rate of prepayments. If the interest rate is

[6]Theoretically, it is possible to expand the state space by the number of tranches in the CMO and implement a recursive, finite-difference solution procedure. Unfortunately, given any reasonable number of tranches (i.e., more than three), the solution procedures become technically infeasible. We thank Scott Richard for pointing out this possibility. See also Kau, Keenan, Muller, and Epperson (1993).

above a critical level, the probability of prepayment in time interval Δt is

$$p_b(t) = 1 - e^{-\lambda(t)\Delta t}. \tag{5}$$

If the interest rate falls below the critical level, the probability of prepayment in the time interval Δt is

$$p_r(t) = 1 - e^{-(\lambda(t)+\rho)\Delta t}. \tag{6}$$

In the limit, as Δt goes to zero, let the continuous prepayment probability per unit of time at any time t for any refinancing cost category i be defined by the function $\pi^i(r, t)$. Hence, the function $\pi^i(r, t)$ is equal to $\pi_b^i(r, t) = \lambda(t)$ when there are only background prepayments and is equal to $\pi_r^i(r, t) = \lambda(t) + \rho$ when the interest rate falls below the critical level $r_c^i(t)$.[7]

Given the critical boundary of interest rates from step one, equations (5) and (6) can be used to determine the prepayments and, therefore, the cash flows at any point in time for any interest rate for mortgagors from refinancing cost category i. CMO valuation requires the allocation of the total cash flows from the underlying collateral to the various tranches. To allocate cash flows at each point in time, it is necessary to retain a memory of prior prepayments. To construct this memory, let $w^i = F^i(0)/F_G(0)$ denote the initial fraction of the pool of mortgages with refinancing cost RF^i, where the subscript G refers to the generic pass-through security (i.e., the overall underlying collateral), let N be the number of refinancing cost categories, let $F_G(0) = \sum_{i=1}^{N} F^i(0)$, and let the fraction of mortgages in category i surviving until time t be $S^i(r, t)$. Thus,

$$dS^i(t) = -\pi^i(t)S^i(t)dt \tag{7}$$

where the dependence of $S^i(t)$ and $\pi^i(t)$ on the interest rate is suppressed for notational convenience.

The allocation of cash flows among the tranches requires the knowledge of the total cash flow from the underlying collateral at each point in time. Let

$$S(t) = \sum_{i=1}^{N} w^i S^i(t) \tag{8}$$

be the fraction of the underlying collateral that survives until time t, and let

$$PR(t) = \sum_{i=1}^{N} w^i \pi^i(t) S^i(t) \tag{9}$$

[7] With these assumptions, the value of a generic MBS can be determined with a finite difference solution procedure with an additional ingredient. Specifically, it is necessary to define the value of the mortgagor's liability and the value of the MBS conditional on the prepayment option remaining unexercised. At each step in the finite difference procedure, the value of the MBS is a weighted average of value of the MBS in the absence of prepayment and the remaining principal balance where the prepayment probability functions in equations (5) or (6) are used to assign the weights.

be the prepayment rate for the declining balance of the underlying collateral at time t. In the absence of prepayments, the cash flow from the underlying collateral is $A = \sum_{i=1}^{N} w^i A^i$ and the remaining principal balance of the underlying collateral is

$$F_G(t) = (A/c)(1 - \exp(-c(T - t))) \tag{10}$$

where c is the coupon rate on the underlying collateral and T is the term to maturity of the loans. Taking into account prepayments, the total cash flows from the underlying collateral to MBS investors is

$$CF_G(t) = S(t)A + PR(t)F_G(t) - S(t)c_s F_G(t). \tag{11}$$

This cash flow is then allocated among the various tranches according to a predetermined hierarchy.

Let the cash flow to tranche Q be $CF_Q(t)$. Then, the value of a CMO tranche, $P(r, S^i, \ldots, S^N, t)$, is the solution to the differential equation

$$1/2 r\sigma(t)^2 P_{rr} + (\phi(t) - \alpha(t)r)P_r + P_t - \sum_{i=1}^{N} \pi^i(t) S^i P_{S^i} + CF_Q - rP = 0, \tag{12}$$

subject to terminal and boundary conditions appropriate to each tranche.[8] The cash flow term, $CF_Q(t)$, is a function of the remaining principal balance of the collateral and the remaining principal balance of tranche Q. The principal balance and the termination date of each tranche are determined simultaneously. The relation between the remaining principal of the underlying collateral, the remaining principal balances of the individual tranches, and the termination date of each tranche can be formulated as a system of simultaneous equations. The system of equations is specific to a given CMO structure and, in general, depends upon the number and types of tranches in the structure (e.g., whether the tranches are sequential or simultaneous pay, whether the structure includes an interest-only or a principal-only tranche, whether the structure contains a planned amortization class-bond or a targeted amortization class-bond, and so on). Additionally, as described in McConnell and Singh (1993), the terminal conditions for each tranche are stochastic and depend upon the particular CMO structure. As an example, determination of the terminal conditions, the remaining principal balance, and the cash flow term for each tranche is outlined in the Appendix for the five-tranche CMO described in Section I.

To summarize, the N differential equations governing mortgagors' liabilities, $V^i(r, t)$ in equation (2a), the equation that governs the value of a CMO tranche, $P(r, t)$, in equation (12), and the N prepayment boundary conditions for the N different refinancing cost categories in equation (3), along with the terminal and natural boundary conditions for each individual tranche comprise our CMO valuation framework.

[8] Henceforth, for notational convenience, the dependence of P on $S^i, \ldots, S^N$ is suppressed.

As noted, the solution of this system of equations is performed in two steps. The N equations representing mortgagors' liabilities are solved first by using the implicit finite difference backward recursion procedure to determine N refinancing boundaries for the different refinancing cost categories of mortgagors. Monte Carlo simulation is then used to solve equation (12) to determine the value of individual CMO tranches. The simulations rely on the cash flow equations and the terminal conditions for each tranche. (These are outlined in the Appendix for the five-tranche example of Section I.) The Monte Carlo procedure is an approximation of the fundamental risk-neutral valuation equation

$$M(r, t) = E_t\left[\int_t^T \left\{CF_Q(s)\exp - \int_t^s r(v)dv\right\}ds\right], \tag{13}$$

where E_t denotes expectations taken with respect to the equivalent Martingale measure of the risk-adjusted process. In this procedure, paths of the risk-adjusted short-term interest rate are generated for the remaining term to maturity of the mortgages. At each point along a path, for each category of refinancing costs, the risk-adjusted interest rate is compared with the critical interest rate. If the interest rate is below the critical level, equation (6) is used to determine the prepayment rate, and if the interest rate is above the critical level, the prepayment rate is given by equation (5). This procedure determines the cash flows along each path of interest rates for mortgages belonging to a particular refinancing cost category. The procedure is repeated for each refinancing cost category. The total cash flow to the tranches at each point in time is then computed as the weighted sum of the cash flows using the fraction of loans surviving, $S^i(t)$, of each refinancing cost category. The allocation of the cash flow among various tranches is determined by the structure of the CMO. The present value of the cash flows along each simulated path is determined by discounting the cash flows at the risk-adjusted interest rate. The entire procedure is repeated for a large number of paths. The value of a CMO tranche is, then, the average of the present values of the cash flows over many iterations of the entire procedure.

III. Implementation of the Dynamic Programming Procedure: Valuation of a Five-tranche CMO

A. Calculation of the Critical Interest Rate Boundary

In this section, we illustrate our valuation procedure with a numerical example. The first step in our procedure depends only upon the model of the term structure of interest rates and the values of the associated parameters, the refinancing cost categories, and the coupon rate and term to maturity of the underlying collateral. To solve the valuation equation for the mortgagor's liability in equation (2a), we use an implicit finite-difference, backward solution procedure with the boundary condition given in equation (3).[9] Starting from the maturity date of the mortgage, the value of the mortgagors' liabilities, $V^i(r, t)$, is compared with the remaining principal balance plus the refinancing cost, $(1 + RF^i)F^i(t)$. If the value exceeds $(1 + RF^i)F^i(t)$, it is set equal to $(1 + RF^i)F^i(t)$. If $V^i(r, t) < (1 + RF^i)F^i(t)$, the value is $V^i(r, t)$. At each point in time, there is a contiguous pair of interest rates such that $V^i(r, t)$ is less than $(1 + RF^i)F^i(t)$ for the higher interest rate and is greater than $(1 + RF^i)F^i(t)$ for the lower interest rate. By interpolation between the two interest rates, we determine the critical interest rate, $r_c^i(t)$. Proceeding backwards in this fashion, the path of this critical prepayment boundary is determined for the refinancing cost category i for a given set of parameters of the term structure.

To illustrate the procedure, we assume the collateral comprises 9.5 percent, 30-year mortgages, and we employ the extended Cox, Ingersoll, and Ross term structure model given in equation (1) with $\phi(t) = 0.02$, $\alpha(t) = 0.2$, and $\sigma(t) = 0.05$. This set of parameters implies that the (asymptotic) infinite maturity zero coupon rate is 9.71 percent. With these parameters, a critical boundary can be determined for each refinancing cost category of mortgages. We assume the collateral is made up of mortgages from five categories of refinancing costs, $RF^i = 0, 3, 6, 9$, and 12 percent for $i = 1, \ldots, 5$.

Figure 1 displays the critical boundary for each of the five refinancing cost categories in our example. Each boundary comprises 360 critical interest rates at monthly intervals. As anticipated, at each point in time, the critical interest rate increases as the refinancing cost decreases. Through time, for the zero refinancing cost category, the critical boundary traces the level of the short-term interest rate that gives the current coupon mortgage rate for a mortgage with a term to maturity equal to the remaining term of the original 9.5 percent mortgage. As the remaining term to maturity declines, the current coupon rate approaches the short-term rate. At maturity, the critical refinancing rate equals the original coupon rate of 9.5 percent.

For mortgages with positive refinancing costs, the critical boundary of interest rates may initially increase. However, as the remaining term to maturity declines, the present value of interest savings due to refinancing becomes smaller because the shorter remaining term to maturity implies that interest must be paid over a shorter time period. For any positive refinancing cost, as the remaining term to maturity approaches zero, the critical interest rate also approaches zero. Hence, for the four categories with positive refinancing costs, the critical boundary eventually declines to zero.

[9]A 360-point grid across time vs. 50 points across interest rates is developed. A transformation of interest rates is performed (see Brennan and Schwartz (1978)) to ensure a high concentration of grid points between 4 and 20 percent for the short-term rate. The transformed interest rate variable is $y = 1/(1 + \gamma r)$. We use a value of 12.5 for γ. The natural boundaries at zero and the critical interest rate, r_c, for the short-term rate are used along with the terminal condition for individual mortgages, $V^i(r, T) = 0$. The Crank-Nicolson finite-difference algorithm is used for solving the equations. To ensure accuracy of the numerical results, we employ an implicit finite-difference procedure.

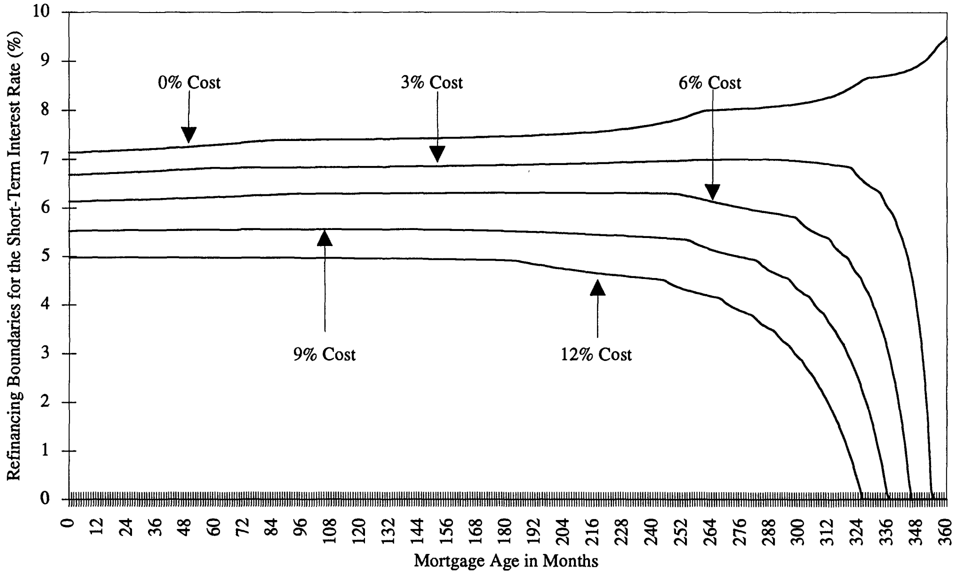

Figure 1. Optimal refinancing boundaries I. This figure displays the short-term interest rate boundaries with five different levels of refinancing costs for the optimal refinancing of 9.5 percent coupon, 30-year, fixed-rate mortgages. Refinancing costs as a percentage of the remaining principal balance for the five categories are (from the highest to the lowest boundary) 0, 3, 6, 9, and 12 percent respectively. The Cox, Ingersoll, and Ross model of the term structure is employed to determine the short-term interest rate boundaries. The parameters of the term-structure model are: $\phi(t) = 0.02$; $\sigma(t) = 0.05$; $\alpha(t) = 0.2$. These parameters imply an asymptotic long-term interest rate of 9.71 percent.

Figure 1 gives the critical path of interest rates for one specification of the parameters of the term structure. A change in the value of any of the parameters of the term structure will result in a different critical prepayment boundary for each level of refinancing costs. Specifically, for the given set of parameters in Figure 1, the implied asymptotic long-term zero coupon rate is 9.71 percent. A change in any of the parameters results in a different implied long-term rate. Keeping the mean reversion parameter $\phi(t)$ and the volatility parameter $\sigma(t)$ constant, an increase in $\alpha(t)$ results in a decline in the asymptotic long-term rate or, equivalently, a decline in the slope of the yield curve. Note, however, that in the Cox, Ingersoll, and Ross model, the initial level of the short-term interest rate does not affect the asymptotic long-term interest rate. In Figure 2, the value of the parameter $\alpha(t)$ is increased to 0.3 from 0.2, which implies a new asymptotic long-term rate of 6.58 percent. At time zero, the refinancing boundaries for each of the refinancing cost categories that are shown in Figure 2 are higher than their corresponding values in Figure 1. A higher refinancing boundary implies a higher rate of prepayments for the 9.5 percent coupon mortgages. As we shall discuss later, an

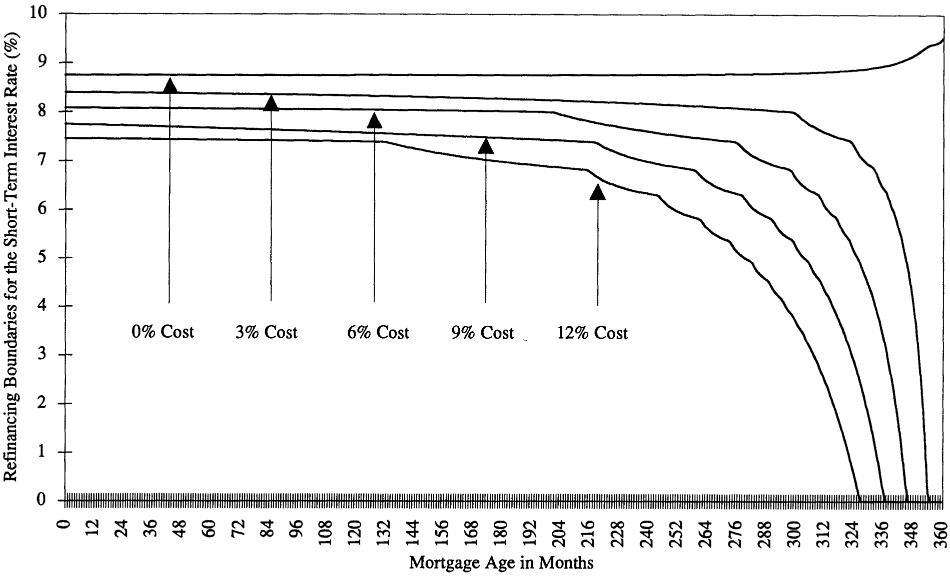

Figure 2. Optimal refinancing boundaries II. This figure displays the short-term interest rate boundaries with five different levels of refinancing costs for the optimal refinancing of 9.5 percent coupon, 30-year, fixed-rate mortgages. Refinancing costs as a percent of the remaining principal balance for the five categories are (from the highest to the lowest boundary) 0, 3, 6, 9, and 12 percent respectively. The Cox, Ingersoll, and Ross model of the term structure is employed to determine the short-term interest rate boundaries. The parameters of the term-structure model are: $\phi(t) = 0.02$; $\sigma(t) = 0.05$; $\alpha(t) = 0.3$. These parameters imply an asymptotic long-term interest rate of 6.58 percent.

increase in $\alpha(t)$ implies a lower discount rate in the second stage of the valuation model. In Section III.C, we conduct a sensitivity analysis with respect to different values of the parameters of the term structure. Although not displayed, each different set of parameters yields a different path of critical interest rates for each refinancing cost category. The interaction of the effect of $\alpha(t)$ on the prepayment boundary and on the discounting of cash flows to the tranches determines the net effect on the value of the tranche.

B. Valuation of a Five-tranche CMO

To implement the second step of our valuation procedure, consider the simple sequential pay five-tranche CMO described in Section I. Let the face values of the three sequential pay tranches and the Z-bond be $F_A(0)$, $F_B(0)$, $F_C(0)$, and $F_Z(0)$. Because the IO pays interest only, it has no stated principal value. Each tranche may have a different coupon rate of interest. Denote these rates as c_A, c_B, c_C, and c_Z. Let c_h be the coupon rate of the tranche paying the highest rate of interest where $h = A$, B, C, or Z. The interest paid to the IO tranche is the difference between the interest payments from the underlying collateral and the promised interest payments to the A, B, C, and Z tranches. Every CMO must satisfy a cash flow requirement and a face value requirement given as

$$F'_G(0) = (A/c_h)(1 - \exp(-c_h T)), \tag{14}$$

$$F_A(0) + F_B(0) + F_C(0) + F_Z(0) \leq \mathrm{Min}[F_G(0), F'_G(0)]. \tag{15}$$

where $F'_G(0)$ is the present value of the scheduled collateral cash flows discounted with the highest coupon rate of interest of any CMO tranche. That is, the promised coupon rates and principal payments must be feasible given the underlying collateral. The condition given in equation (14) insures that the interest payments promised to each tranche can be met regardless of the prepayment rate of the underlying collateral. The condition given in equation (15) insures that the face value of the collateral is at least equal to the sum of the face values of the individual tranches. Enforcing these conditions can lead to "overcollateralization." In that case, the "excess" collateral may be held by the issuer as equity interest or used to support a principal only tranche.

Because prepayments are uncertain, the maturities of the tranches are stochastic and depend upon the rate of mortgage prepayments. Let t_1, t_2, t_3, and t_4 denote the times at which the A, B, C, and Z tranches are retired, and, if the CMO is overcollateralized, let t_5 be the time at which the underlying collateral is finally retired. For purposes of developing the model, the cash flows to the four tranches can be allocated into time intervals demarcated by the dates at which retirement of the tranches occurs. The first time interval is denoted as $0 \leq t \leq t_1$, the second is $t_1 < t \leq t_2$, the third is $t_2 < t \leq t_3$, the fourth is $t_3 < t \leq t_4$, and the fifth is $t_4 \leq t \leq t_5$. When the CMO is not overcollateralized, $t_4 = t_5$.

The cash flows from the underlying collateral consist of the scheduled principal and interest payments from the surviving mortgages plus any prepayments. The prepayment rate for each refinancing cost category is determined by the critical interest rate boundary for that category and is expressed as $\pi^i(r, t) = \pi^i(r, t \mid r_c^i(t))$. Each time the interest rate falls below the critical level, the prepayment rate is $\pi_r^i(r, t) = \lambda(t) + \rho$. The prepayment rate for background prepayments is $\pi_b^i(r, t) = \lambda(t)$. The fraction of loans from cost category i surviving until time t is determined by equation (7). The fraction of the total underlying collateral surviving until time t is determined by equation (8), and the prepayment rate for the total underlying collateral is determined with equation (9). These are then used with equation (11) to determine the total cash flow from the underlying collateral. The Appendix presents the equations that allocate the cash flow among the various tranches at each point in time. The cash flow equation for each tranche along with equation (12) provide the framework for valuing each tranche separately. The valuation is accomplished by using the risk-adjusted discounting procedure of equation (13).

C. Numerical Example

Table I presents the results of a numerical illustration for a five-tranche CMO with the Cox, Ingersoll, and Ross model of the term structure. In this illustration, the short-term interest rate, $r(0)$, and the drift parameter, $\alpha(t)$, are varied to obtain different shapes and levels of the term structure. The values of the five tranches and the generic MBS are presented for three different levels of the short-term interest rate and five values of the parameter $\alpha(t)$. The three levels of the short-term rate are 3, 6, and 9 percent. The implied long-term interest rate is defined as the 30-year zero coupon rate corresponding to a given level of the drift parameter $\alpha(t)$. Thus, the results are generated for 15 different yield curves.

The coupon rate of the underlying collateral is 9.5 percent with a 0.5 percent servicing and guarantee fee, which makes the pass-through coupon rate of the MBS that collateralizes the CMO equal to 9.0 percent. The A, B, and C tranches each have initial principal equal to 30 percent of the principal of underlying collateral and the Z-bond makes up the remaining 10 percent. The IO strip is based on the interest differential between the collateral pass-through coupon rate and the coupon rates of the four other tranches. The collateral consists of mortgages from five refinancing cost categories each of which comprises 20 percent of the initial value of the underlying mortgage collateral. The proportional cost for the five categories are 0, 3, 6, 9, and 12 percent, respectively. The prepayment delay parameter, ρ, is equal to 2.0, which implies that mortgagors evaluate their refinancing alternatives every six months. The background level of prepayments is assumed to be 5 percent per year.

The values of the various tranches are displayed in columns 4 through 8 of Table I. Column 9 gives the value of the generic MBSs that collateralize the CMO. The columns labeled A, B, C, and Z express the values of the CMO tranches as a percentage of the face value of the relevant tranche. Column 8 gives the value of the IO as a percentage of the face value of the collateral. Thus, in every case, the value of the collateral is merely the value of the IO plus the weighted sum of the values of the other tranches.

Because the short-term rate determines the level of the entire term structure in a single-factor model, the values of the generic MBSs and the individual tranches are quite sensitive to changes in the short-term rate. They are also sensitive to changes in the value of the parameter $\alpha(t)$. For any given level of the short-term rate, an increase in $\alpha(t)$ implies a lower long-term interest rate.

Two opposing interest rate effects are at work in determining the pattern in the tranche values. With a decline in the long-term rate, the discounting effect tends to increase tranche value. However, a decrease in rates tends to increase the probability of refinancing. Depending upon which effect predominates, an increase in $\alpha(t)$ may lead to either an increase or a decrease in the value of the generic CMO. In the top two sections of Table I, a decrease in $\alpha(t)$ first leads to an increase in the value of the generic CMO and then a

Table I

Valuation of a Five-tranche Sequential Pay CMO with the Cox, Ingersoll, and Ross Model of the Term Structure of Interest Rates

Short-Term Interest Rate (%)	$\alpha(t)$	Implied Long-Term Interest Rate (%)	A Tranche	B Tranche	C Tranche	Z Bond	IO	Generic MBS
3.0	0.30	6.2	100.62	101.81	103.72	107.69	0.32	102.94
3.0	0.25	7.2	100.62	101.80	103.72	107.69	0.32	102.94
3.0	0.20	8.7	100.62	101.80	103.72	107.84	0.38	103.01
3.0	0.15	10.7	100.88	104.24	109.44	93.81	1.84	105.59
3.0	0.10	13.9	106.38	106.94	98.48	54.65	3.72	102.72
6.0	0.30	6.5	100.27	100.88	101.87	104.15	0.33	101.65
6.0	0.25	7.6	100.27	100.89	101.86	103.95	0.43	101.73
6.0	0.20	9.1	100.46	101.85	103.05	99.85	1.67	103.26
6.0	0.15	14.7	100.89	95.77	86.83	43.83	3.39	92.82
6.0	0.10	14.7	100.89	95.77	86.83	43.83	3.39	92.82
9.0	0.30	6.8	99.32	99.57	99.98	101.42	1.04	100.85
9.0	0.25	8.0	98.78	98.78	98.84	99.14	1.82	100.66
9.0	0.20	9.6	97.96	96.32	94.80	86.01	3.05	98.38
9.0	0.15	11.9	97.03	91.86	86.47	59.38	3.32	91.87
9.0	0.10	15.6	95.85	86.16	77.05	35.29	3.11	84.36

Collateral attributes: Coupon rate = 9.5 percent; maturity = 360 months; servicing and guarantee fee = 0.5 percent.

Number of refinancing cost categories: 5 categories in equal proportions of the underlying collateral.

Refinancing costs as a percentage of remaining principal for 5 categories: 0, 3, 6, 9, and 12 percent.

Prepayment delay parameter: $\rho = 2.0$.

Face value of tranche/face value of collateral: A = 0.30; B = 0.30; C = 0.30; and Z = 0.10.

Coupon rates of tranches: A = 8.0 percent; B = 8.5 percent; C = 8.5 percent; and Z = 8.5 percent.

Parameters of the Cox, Ingersoll, and Ross model of the term structure: $\phi(t) = 0.02$; $\sigma(t) = 0.05$.

decrease in value. The way in which the change in the value of the generic MBS is allocated across the CMO tranches depends on the initial level of the short-term rate and on the maturity (and priority) of the tranche. In general, longer maturity tranches and the IO are more sensitive to changes in the level and shape of the term structure than are the shorter maturity tranches. That, of course, is not news. What is new, at least four purposes, is that our procedure for CMO tranche valuation gives results that are consistent with intuition and with empirically based models of CMO analysis.

D. Alternative Models of the Term Structure of Interest Rates

As we noted, our procedure for analyzing CMOs can be employed with any arbitrage-free model of the term structure. Hull and White (1990) examine the sensitivity of the valuation of interest-rate-dependent derivative securities to two alternative characterizations of the term structure—the extended Cox, Ingersoll, and Ross model and the extended version of the Vasicek (1977) model. They analyze bond options and interest rate caps with maturities up to five years and conclude that the values of these securities are not sensitive to the particulars of the term-structure model employed so long as the model is consistent with the entire observed yield curve. Their conclusions may or may not hold for the long-dated options embedded in 30-year MBSs. Furthermore, even if the alternative models yield similar values for generic MBSs, it is possible that the models will produce different values for the individual tranches because the reallocation of call risk among the tranches may depend on the characteristics of the specific term-structure model employed.

In the spirit of Hull and White, we analyze the sensitivity of CMO tranche values to three models of the term structure—the extended version of the Cox, Ingersoll, and Ross model, the extended version of the Vasicek model, and the Heath, Jarrow, and Morton (1992) model—all three of which can be made consistent with the entire observed yield curve.[10] We begin our analysis with the yield curves that are generated with the Cox, Ingersoll, and Ross model used for the valuation results in Table I. Given the yield curves of Table I, we derive the implied time-dependent parameters of the risk-adjusted interest rate process of the Vasicek model

$$dr = [\phi(t) - \alpha(t)r]dt + \sigma(t)dt. \tag{16}$$

In the Vasicek model (in contrast to the Cox, Ingersoll, and Ross model), short-term interest rate volatility is independent of the level of the short-term interest rate. To derive the yield curves of the Vasicek model, we employ the procedure used in Hull and White (1990). In their framework, the initial term structure of interest rates is represented by the price of a pure discount bond with maturity T, $A(0, T)\exp - B(0, T)r(0)$, where $r(0)$ is the initial short-

[10] The Heath, Jarrow, and Morton (1992) model was not available when Hull and White (1990) conducted their analysis.

term interest rate and $A(0, T)$ and $B(0, T)$ are functions of maturity. The parameters of the Cox, Ingersoll, and Ross model are used to determine the value of $A(0, T)$ and $B(0, T)$ (equations (28) and (29) in Hull and White) as

$$A(0, T) = \left[\frac{2\gamma e^{(\gamma+\alpha)T/2}}{(\gamma+\alpha)(e^{\gamma T}-1)+2\gamma}\right]^{2\phi/\sigma^2}$$

and

$$B(0, T) = \left[\frac{2(e^{\gamma T}-1)}{(\gamma+\alpha)(e^{\gamma T}-1)+2\gamma}\right],$$

where $\gamma = \sqrt{\alpha^2 + 2\sigma^2}$. Equations (13), (14), (15), and (16) from Hull and White are then used to determine the time-varying parameters such that the Vasicek model fits the yield curve generated with the Cox, Ingersoll, and Ross model. Equations (13) and (14), respectively, are

$$B(t, T) = [B(0, T) - B(0, t)]/B_t(0, t)$$

and

$$A'(t, T) = A'(0, T) - A'(0, t) - B(t, T)A'_t(0, t)$$
$$- 1/2[B(t, T)B_t(0, t)]^2 \int_0^t \left[\sigma(\tau)/B_\tau(0, \tau)^2\right] d\tau,$$

where a subscript denotes a derivative with respect to that variable and $A'(t, T) = \log[A(t, T)]$. The drift parameters of the Vasicek model are obtained from equations (15) and (16) in Hull and White as

$$\alpha(t) = -B_{tt}(0, t)/B_t(0, t),$$

and

$$\phi(t) = -\alpha(t)A'_t(0, t) - A'_{tt}(0, t) + [B_t(0, t)]^2 \int_0^t [\sigma(\tau)/B_\tau(0, \tau)]^2 d\tau.$$

With the Vasicek model, the differential equation governing the value of a mortgagor's liability is

$$1/2\sigma(t)^2 V^i_{rr} + (\phi(t) - \alpha(t)r)V^i_r + V^i_t + A^i - rV^i = 0, \qquad i = 1, \ldots, N \tag{17}$$

where i represents the refinancing cost category i. The differential equation governing the value of CMO tranche Q is

$$1/2\sigma(t)^2 P_{rr} + (\phi(t) - \alpha(t)r)P_r + P_t - \sum_{i=1}^{N} \pi^i(t)S^i P_{S^i} + CF_Q - rP = 0. \tag{18}$$

The cash flow term, CF_Q, and boundary conditions are the same as with the Cox, Ingersoll, and Ross model. The CMO valuation procedure is the same as outlined earlier.

The Heath, Jarrow, and Morton framework, which utilizes the entire term structure of forward rates as a starting point, permits several characterizations of the term structure of volatilities. For comparison purposes, we use a one-factor Heath, Jarrow, and Morton model and assume a term structure of forward rate volatilities that is consistent with the Vasicek model. In the Heath, Jarrow, and Morton model, the infinitesimal change in the instantaneous forward rate $df(t, T)$ at time t for the future time T is

$$df(t, T) = a(t, T)dt + \sigma e^{(-\mu/2)(T-t)}dz \tag{19}$$

where $a(t, T)$ is a time-dependent drift parameter, and the parameters σ and μ determine the structure of forward-rate volatilities. The evolution of the short-term interest rate that is consistent with the forward-rate dynamics is given as

$$r(t) = f(0, t) - 2(\sigma/\mu)^2[(1 - e^{-\mu t}) - 2(1 - e^{-(\mu/2)t})] + \int_0^t \sigma e^{-(\mu/2)(t-v)}dz(v) \tag{20}$$

where $f(0, T)$ is the initial term structure of forward rates. With this specification of forward-rate dynamics, the short-term interest rate follows a mean reverting process given by

$$dr = \mu/2[\eta(t) - r]dt + \sigma dz \tag{21}$$

where

$$\eta(t) = (2/\mu)\,\partial f(0, t)/\partial t + f(0, t) + 2(\sigma/\mu)^2[1 - e^{-\mu t}].$$

With this characterization of the Heath, Jarrow, and Morton model, equations (17) and (18) can be used to value CMO tranches with two modifications: (1) $\phi(t)$ is replaced with $(\mu/2)\eta(t)$ and (2) $\alpha(t)$ is replaced with $\mu/2$.

Tranche values with the two alternative term-structure models are given in Table II. Panel A shows the tranche values obtained by using the extended Vasicek model. The volatility parameter, $\sigma(t)$, in the Vasicek model is $0.05\sqrt{r(0)}$, where $r(0)$ is the initial short-term interest rate. This makes the volatility of the short-term interest rate identical to the initial volatility in the Cox, Ingersoll, and Ross model. In the Heath, Jarrow, and Morton model, the constant parameter μ determines the term structure of volatility, but does not influence the initial term structure of interest rates. In the Heath, Jarrow, and Morton model, the parameter μ also governs the rate of mean reversion of the short-term interest rate. We set μ equal to 2α where α is the parameter governing the rate of mean reversion in the Cox, Ingersoll, and Ross model, so that the rate of mean reversion is equivalent across the two models.

Table II presents values of the generic MBS and the five tranches of our example CMO for the 15 interest rate scenarios of Table I. Perhaps surprisingly, a comparison of the two panels in Table II with Table I shows that, so long as the specification of the term-structure model is consistent with the

Table II

Valuation of a Five-tranche Sequential Pay CMO with the Vasicek and Heath, Jarrow, and Morton Models of the Term Structure of Interest Rates

Panel A: Vasicek term structure model with Cox, Ingersoll, and Ross parameters

Short-Term Interest Rate (%)	$\alpha(t)$	Implied Long-Term Interest Rate (%)	A Tranche	B Tranche	C Tranche	Z Bond	IO	Generic MBS
3.0	0.30	6.2	100.62	101.81	103.72	107.67	0.32	102.93
3.0	0.25	7.2	100.62	101.81	103.72	107.67	0.32	102.93
3.0	0.20	8.7	100.62	101.80	103.69	107.76	0.35	102.96
3.0	0.15	10.7	100.87	104.14	109.72	94.02	1.83	105.65
3.0	0.10	13.9	106.38	106.96	98.57	55.01	3.72	102.79
6.0	0.30	6.5	100.27	100.88	101.86	104.12	0.33	101.64
6.0	0.25	7.6	100.27	100.88	101.85	104.01	0.40	101.70
6.0	0.20	9.2	100.42	101.80	103.23	100.12	1.64	103.29
6.0	0.15	11.4	101.64	100.85	96.17	70.36	3.52	100.16
6.0	0.10	14.8	100.88	95.74	86.82	43.91	3.39	92.81
9.0	0.30	6.9	99.31	99.57	99.98	101.46	1.09	100.89
9.0	0.25	8.1	98.74	98.72	98.76	99.04	1.94	100.71
9.0	0.20	9.7	97.93	96.21	94.52	84.98	3.12	98.22
9.0	0.15	12.0	97.00	91.70	86.27	58.93	3.31	91.70
9.0	0.10	15.7	95.83	86.06	76.93	35.12	3.11	84.26

Table II—*Continued*

Panel B: Heath, Jarrow, and Morton term-structure model with Cox, Ingersoll, and Ross parameters

Short-Term Interest Rate (%)	$\alpha(t)$	Implied Long-Term Interest Rate (%)	A Tranche	B Tranche	C Tranche	Z Bond	IO	Generic MBS
3.0	0.60	6.2	100.62	101.80	103.70	107.62	0.32	102.91
3.0	0.50	7.2	100.62	101.80	103.70	107.62	0.32	102.91
3.0	0.40	8.7	100.62	101.79	103.65	107.47	0.33	102.89
3.0	0.30	10.7	100.79	103.56	111.05	93.73	1.82	105.82
3.0	0.20	13.9	106.33	106.76	98.36	54.78	3.72	102.63
6.0	0.60	6.5	100.27	100.86	101.76	103.77	0.32	101.56
6.0	0.50	7.6	100.27	100.86	101.78	103.72	0.32	101.56
6.0	0.40	9.2	100.31	101.17	104.17	103.78	1.28	103.35
6.0	0.30	11.4	101.82	101.07	96.33	70.69	3.62	100.45
6.0	0.20	14.8	100.83	95.74	86.91	44.26	3.40	92.86
9.0	0.60	6.9	99.86	100.01	100.13	100.68	0.43	100.50
9.0	0.50	8.1	99.83	99.99	100.21	103.25	0.99	101.33
9.0	0.40	9.7	98.18	97.31	95.75	87.33	3.56	99.66
9.0	0.30	12.0	97.04	92.01	86.74	60.05	3.34	92.08
9.0	0.20	15.7	95.82	86.21	77.23	35.84	3.11	84.48

Collateral attributes: Coupon rate = 9.5 percent; maturity = 360 months; servicing and guarantee fee = 0.5 percent.
Number of refinancing cost categories: 5 categories in equal proportions of underlying collateral.
Refinancing costs as a percentage of face value for 5 categories: 0, 3, 6, 9, and 12 percent.
Prepayment delay parameter: $\rho = 2.0$.
Face value of tranche/face value of collateral: A = 0.30; B = 0.30; C = 0.30; and Z = 0.10.
Coupon rates of tranches: A = 8.0 percent; B = 8.5 percent; C = 8.5 percent; and Z = 8.5 percent.
Parameters of the Cox, Ingersoll, and Ross model of the term structure: $\phi(t) = 0.02$; $\sigma(t) = 0.05$.

observed yield curve, not only are the values of generic MBSs relatively insensitive to the particular one-factor model of the term-structure model employed, but so too are the values of the individual tranches. Thus, our CMO valuation procedure appears to be robust to alternative specifications of a single-factor model of the term structure.

E. Alternative Specifications of the Refinancing Cost Categories

Given that the choice of the term-structure model employed appears to be of relatively little consequence in the valuation of CMO tranches, the next question becomes whether the values are equally insensitive to alternative characterizations of prepayment behavior. To address this question, we value the CMO under two different characterizations of the distribution of refinancing costs and with three different specifications of the prepayment delay parameter, ρ. The results are presented in Table III. Panel A of the table assumes equal proportions of mortgagors with refinancing costs of 3, 6, 9, 12, and 15 percent. Panel B assumes equal proportions of mortgagors with refinancing costs of 1, 2, 10, 20, and 30 percent. In each panel, the delay parameter, ρ, is set at three different levels—1, 2, and 12—that correspond to mortgagors evaluating their refinancing alternatives at one-year, six-month, and one-month intervals. Both panels assume a short-term interest rate of 6 percent and employ the five different levels of the parameter $\alpha(t)$ used in Table I. The tranche values of Table III can be compared with the tranche values in the middle section of Table I.

In general, the results are compatible with intuition. Overall, differences in the prepayment delay parameter and refinancing costs are likely to have little effect on tranche valuation when the probability of prepayment is low. Only when prepayment is a serious possibility are differences in refinancing costs and prepayment delay likely to have any impact. This result is apparent in the scenarios in which the implied long-term rate is high. For example, when the long-term rate is 14.7 percent the value of the generic MBS in Panel A, Panel B, and the middle section of Table I is 92.82 regardless of the prepayment-delay parameter. Similarly, the values of the various tranches are identical for these scenarios. In the scenarios in which the probability of prepayment is higher (i.e., when the implied long-term interest rate is lower), the effect of differences in the cost of refinancing and prepayment delay is more pronounced. For example, consider the three cases in Panel A in which the implied long-term interest rate is 6.5 percent and the prepayment delay parameters are 1, 2, and 12 (i.e., refinancing decisions are made at one-year, six-month, and one-month intervals). The value of the generic MBS declines from 103.03 to 101.68 to 100.45 as the interval for refinancing decisions declines. Similarly, the values of each of the tranches declines as the delay parameter decreases, with the greatest effect being manifest in the tranches with the longer maturities. For example, the value of the Z-bond declines from 108.00 to 100.44 as the refinancing interval declines from one-year to one-month.

Panels A and B of Table III, along with the middle section of Table I, can be compared to examine the sensitivity of tranche valuation to specification of the costs of refinancing. The effect of some changes in the distributions of refinancing costs are straightforward, but not others. For example, in Table I, the refinancing cost categories are 0, 3, 6, 9, and 12 percent in equal proportions, whereas in Panel A of Table III, the categories are 3, 6, 9, 12, and 15 percent. In essence, the costs are uniformly higher in Panel A of Table III in comparison with those in Table I. As expected, the value of each tranche is lower in Table III than the comparable tranche in Table I. In Panel B of Table III, the refinancing cost categories are "stretched" relative to those in Table I. In Table III, the cost categories are 1, 2, 10, 20, and 30 percent. The values of the tranches here are also higher than those of the comparable tranches in Table I, but the greatest effect occurs in the longer maturity tranches and the IO.

One other comparison is useful to consider. Suppose two different combinations of the delay parameter and the refinancing cost distributions give rise to the same or a similar value of the underlying generic MBSs. Does that imply that the value of the tranches will be the same? As an example, consider the case in Panel A where the long-term interest rate is 6.52 percent and the prepayment delay parameter is 1. Compare that with the case in Panel B in which the long-term rate is again 6.52 percent and the delay parameter is 2. The value of the generic MBSs in the two panels are nearly identical—103.03 vs. 103.12. However, the Panel B value of the Z-bond exceeds the Panel A value by 3.54, and the Panel B values of the A, B, and C tranches are less than the corresponding values in Panel A by 0.10, 0.46, and 0.50, respectively. Hence, for any given term structure, a specification of the prepayment parameters that leads to a correct value of the generic MBS prices may lead to incorrect valuation of CMO tranches. As might be anticipated from the discussion above, this problem is most severe when prepayment probabilities are high. The important message here is that use of the rational valuation method for CMO valuation requires careful (and accurate) estimation of the parameters of the refinancing cost categories and the prepayment-delay parameter. Stanton's (1993a, 1993b) work in that regard deserves thoughtful consideration and further development.

IV. Summary and Conclusions

CMOs have come to dominate the mortgage-backed securities market. Because CMO tranches rarely trade as generic instruments, "mortgage analytics" play a major role in the trading of CMO tranches. We propose a two-step procedure for analyzing CMO tranches that is in the spirit of a dynamic programming problem. The novelty of our procedure is that we employ a rational prepayment valuation model in step one that determines a critical level of interest rates at which value-minimizing mortgagors would optimally refinance their mortgages. The rational valuation model is solved

Table III

Valuation of Five-tranche Sequential Pay CMO with the Cox, Ingersoll, and Ross Model of the Term Structure of Interest Rates and Various Specifications of Refinancing Cost Categories and Prepayment Delays

Panel A: Refinancing Costs as a Percentage of Face Value for 5 Categories of Refinancing Costs: 3, 6, 9, 12, and 15 Percent

Prepayment Delay Parameter	$\alpha(t)$	Implied Long-Term Interest Rate (%)	A Tranche	B Tranche	C Tranche	Z Bond	IO	Generic MBS
1.0	0.30	6.5	100.42	101.58	103.42	108.00	0.60	103.03
1.0	0.25	7.6	100.43	101.66	103.52	107.89	0.89	103.36
1.0	0.20	9.1	100.99	103.47	104.08	99.52	2.55	105.06
1.0	0.15	11.3	101.81	100.99	96.16	70.14	3.59	100.29
1.0	0.10	14.7	100.89	95.77	86.83	43.83	3.39	92.82
2.0	0.30	6.5	100.27	100.88	101.89	104.29	0.34	101.68
2.0	0.25	7.6	100.27	100.92	102.05	104.71	0.51	101.95
2.0	0.20	9.1	100.67	102.85	103.82	99.70	2.20	104.37
2.0	0.15	11.3	101.79	100.97	96.16	70.18	3.59	100.28
2.0	0.10	14.7	100.89	95.77	86.83	43.83	3.39	92.82
12.0	0.30	6.5	100.18	100.22	100.44	100.83	0.12	100.45
12.0	0.25	7.6	100.18	100.22	100.44	101.02	0.14	100.49
12.0	0.20	9.1	100.21	101.67	103.54	99.92	1.66	103.28
12.0	0.15	11.3	101.74	100.92	96.18	70.29	3.56	100.25
12.0	0.10	14.7	100.89	95.77	86.83	43.83	3.39	92.82

Table III—*Continued*

Panel B: Refinancing Costs as a Percentage of Face Value for 5 Categories of Refinancing Costs: 1, 2, 10, 20, and 30 Percent

Prepayment Delay Parameter	$\alpha(t)$	Implied Long-Term Interest Rate (%)	A Tranche	B Tranche	C Tranche	Z Bond	IO	Generic MBS
1.0	0.30	6.5	100.51	101.97	104.64	116.05	0.96	104.70
1.0	0.25	7.6	100.63	102.61	106.37	119.02	1.68	106.46
1.0	0.20	9.1	100.87	103.47	104.71	100.26	2.47	105.22
1.0	0.15	11.3	101.73	100.92	96.20	70.35	3.55	100.24
1.0	0.10	14.7	100.89	95.77	86.83	43.83	3.39	92.82
2.0	0.30	6.5	100.32	101.12	102.92	111.54	0.66	103.12
2.0	0.25	7.6	100.39	101.62	105.24	118.24	1.41	105.40
2.0	0.20	9.1	100.56	102.81	104.74	100.67	2.20	104.70
2.0	0.15	11.3	101.68	100.86	96.24	70.49	3.53	100.21
2.0	0.10	14.7	100.89	95.77	86.83	43.83	3.39	92.82
12.0	0.30	6.5	100.18	100.29	101.17	106.09	0.34	101.44
12.0	0.25	7.6	100.18	100.44	103.81	117.13	1.11	104.15
12.0	0.20	9.1	100.20	101.85	104.72	101.13	1.88	104.02
12.0	0.15	11.3	101.60	100.72	96.35	70.81	3.47	100.15
12.0	0.10	14.7	100.89	95.77	86.83	43.83	3.39	92.82

Collateral attributes: Coupon rate = 9.5 percent; maturity = 360 months; servicing and guarantee fee = 0.5 percent.
Number of refinancing cost categories: 5 categories in equal proportions of the underlying collateral.
Face value of tranche/face value of collateral: A = 0.30; B = 0.30; C = 0.30; and Z = 0.10.
Coupon rates of tranches: A = 8.0 percent; B = 8.5 percent; C = 8.5 percent; and Z = 8.5 percent.
Parameters of the Cox, Ingersoll, and Ross model of the term structure: $\phi(t) = 0.02$; $\sigma(t) = 0.05$.
Short-term interest rate: $r(0) = 6.0$ percent.

by means of an implicit finite difference solution procedure. The critical level of interest rates depends upon the remaining term to maturity of the mortgage and the refinancing cost confronted by the mortgagor. Given several categories of refinancing cost mortgagors, a critical path of interest rates is determined for each refinancing cost category. These critical paths are then used in the second step of the procedure, along with Monte Carlo simulation, to determine the cash flows to and the value of each CMO tranche.

Sensitivity analysis indicates that the procedure yields results consistent with intuition and with the pattern of observed market prices. The analysis also indicates that the results are generally insensitive to the particular single-factor model of the term structure employed. The results are, however, sensitive to assumptions about refinancing costs confronted by mortgagors and to other aspects of the refinancing decision. As always, with mortgage valuation, the results point to the importance of accurate estimation of the parameters of the prepayment process.

Appendix

The remaining principal balances and the retirement dates of the tranches in the five-tranche CMO structure outlined in Section III.*A* depend on the rate at which the underlying collateral is retired. It is assumed that tranches A, B, and C are retired sequentially, while the Z tranche accrues interest. The Z tranche starts receiving cash flows only after the three prior tranches are fully retired. The IO tranche, however, receives payments throughout the life of the collateral. The remaining balance of the underlying collateral is a function of path of interest rates since the origination of the mortgage pool and is measured by the fraction of the pool surviving until time t, $S(t)$. The cash flow, $CF_Q(t)$, to any tranche, Q, at time, t, depends on the remaining principal balance of the tranche at time t and is contingent on the path of the interest rate from the origination of the collateral. The cash flows and the remaining principal balances of the various tranches are determined from a set of simultaneous equations for one specific realization of the interest rate path. The retirement dates of each tranche are determined from the cumulative principal paid on the underlying collateral.

Time t_1, the retirement date of the A tranche, occurs when the following equation is satisfied:

$$F_A = F_G(0) - S(t_1)F_G(t_1) + F_Z(0)(\exp(c_Z t_1) - 1). \tag{A1}$$

The first two terms on the right give the total principal paid on the underlying collateral. The last term gives the accrued interest on the Z-bond that is used to retire A tranche. Thus, time t_1 occurs when the A tranche is fully retired.

To determine the cash flow rate to A tranche at any time t, it is necessary to determine the (growing) principal of the Z-bond at t and the remaining principal of A tranche. At any time prior to t_3 (when the Z-bond begins to receive principal payments), the principal of the Z-bond is

$$F_Z(t) = F_Z(0)\exp(c_Z t). \tag{A2}$$

The remaining principal of the A tranche at any time up to and including t_1 is

$$F_A(t) = F_A(0) - [\{F_G(0) - S(t)F_G(t)\} + F_Z(0)\{\exp(c_Z t) - 1\}]. \tag{A3}$$

The cash flow rate to the investor in tranche A is

$$CF_A(t) = S(t)[A - cF_G(t)] + PR(t)F_G(t) + c_Z F_Z(t) + c_A F_A(t). \tag{A4}$$

The first term on the right represents the scheduled principal payments from the collateral, the second term gives the cash flows from prepayments on the collateral, the third term is the interest accrued on the Z-bond, and the fourth term is the interest paid on the remaining principal of the A tranche.

The cash flows to the other tranches during the first time interval, $0 \le t \le t_1$, are $CF_B(t) = c_B F_B(0)$; $CF_C(t) = c_C F_C(0)$; $CF_Z(t) = 0$; and $CF_{IO}(t) = CF_G(t) - CF_A(t) - CF_B(t) - CF_C(t)$. That is, the cash flow to the A and B tranches is just the promised interest rate times the initial face value, the cash flow from the Z-bond is zero, and the cash flow to the IO is just the difference between the cash flow to the collateral and the cash flows to the other tranches.

Time t_2 occurs when the following equation is satisfied

$$F_A(0) + F_B(0) = F_G(0) - S(t_2)F_G(t_2) + F_Z(0)[\exp(c_Z t_2) - 1]. \tag{A5}$$

That is, t_2 occurs when the B tranche is fully retired. During the interval $t_1 < t \le t_2$, the remaining principal balance on the B tranche is

$$F_B(t) = F_B(0) - [\{F_G(0) - S(t)F_G(t)\} + F_Z(0)\{\exp(c_Z t) - 1\} - F_A(0)] \tag{A6}$$

and the cash flow rate to the B tranche is

$$CF_B(t) = S(t)[A - cF_G(t)] + PR(t)F_G(t) + c_Z F_Z(t) + c_B F_B(t). \tag{A7}$$

Terms on the right in equation (A7) are analogous to those in equation (A4). The cash flows to the other sequential tranches and the IO during the second time interval, $t_1 < t \le t_2$, are $CF_A(t) = CF_Z(t) = 0$, $CF_C(t) = c_C F_C(0)$; and $CF_{IO}(t) = CF_G(t) - CF_B(t) - CF_C(t)$.

Time t_3 occurs when the B tranche is fully retired. That is, when

$$F_A(0) + F_B(0) + F_C(0) = F_G(0) - S(t_3)F_G(t_3) + F_Z(0)[\exp(c_Z t_3) - 1]. \tag{A8}$$

During the third interval, $t_2 < t \le t_3$, the remaining principal on the C tranche is

$$F_C(t) = F_C(0) - [\{F_G(0) - S(t)F_G(t)\} + F_Z(0)\{\exp(c_Z t) - 1\} - F_A(0) - F_B(0)] \tag{A9}$$

and the cash flow rate to the C tranche is

$$CF_C = S(t)[A - cF_G(t)] + PR(t)F_G(t) + c_Z F_Z(t) + c_C F_C(t). \tag{A10}$$

During this interval, the cash flows to the A and B tranches and the Z-bond are zero, and the cash flow to the IO is $CF_{IO}(t) = CF_G(t) - CF_C(t)$.

Time t_4 occurs when the Z-bond is fully retired,

$$F_A(0) + F_B(0) + F_C(0) + F_Z(0) = F_G(0) - S(t_4)F_G(t_4). \tag{A11}$$

The remaining principal balance of the Z-bond at time t, where $t_3 < t \leq t_4$, is

$$F_Z(t) = F_A(0) + F_B(0) + F_C(0) + F_Z(0) - F_G(0) + S(t)F_G(t) \tag{A12}$$

and the cash flow rate to the Z-bond is

$$CF_Z(t) = S(t)[A - cF_G(t)] + PR(t)F_G(t) + c_Z F_Z(t). \tag{A13}$$

The cash flows to the A, B, and C tranches during $t_3 < t \leq t_4$ are zero and $CF_{IO}(t) = CF_G(t) - CF_Z(t)$.

When the CMO is not overcollateralized, $t_4 = t_5$, and the CMO is fully retired at t_4. When the CMO is overcollateralized, t_5 occurs when $S(t_5)F_G(t_5) = 0$. That is, t_5 occurs when the underlying collateral is fully retired. During the interval $t_4 < t \leq t_5$, the cash flow is divided between the IO and the equity interest in the overall CMO structure due to overcollateralization, and the cash flows to the other tranches are zero. In our exposition, we assume no overcollateralization.

Let $P_Q(t)$ denote the value of any CMO tranche, Q, at time, t. The terminal conditions for the solution of the differential equation (12) for the individual securities are

$$P_G(t_5) = P_A(t_1) = P_B(t_2) = P_C(t_3) = P_Z(t_4) = P_{IO}(t_5) = 0. \tag{A14}$$

REFERENCES

Brennan, M. J. and E. S. Schwartz, 1978, Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis, *Journal of Financial and Quantitative Analysis* 13, 461–474.

———, 1982, An equilibrium model of bond pricing and a test of market efficiency, *Journal of Financial and Quantitative Analysis* 3, 301–329.

Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–407.

Dunn, K. B. and J. J. McConnell, 1981a, Valuation of GNMA mortgage-backed securities, *Journal of Finance* 36, 599–617.

———, 1981b. A comparison of alternative models for pricing GNMA mortgage-backed securities, *Journal of Finance* 36, 375–392.

Dunn, K. B. and C. S. Spatt, 1986. The effect of refinancing costs and market imperfections on the optimal call strategy and the pricing of debt contracts, Unpublished manuscript, Carnegie-Mellon University.

Heath, D., R. A. Jarrow, and A. S. Morton, 1992, Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation, *Econometrica* 60, 77–105.

Ho, T. S. Y. and S.-B. Lee, 1986, Term structure movements and pricing interest rate contingent claims, *Journal of Finance* 41, 1011–1028.

Hull, J. C. and A. D. White, 1990, Pricing interest rate derivative securities, *The Review of Financial Studies* 3, 573–592.

Johnston, E. T. and L. D. Van Drunen, 1988, Pricing mortgage pools with heterogeneous mortgagors: Empirical evidence, Unpublished manuscript, University of Utah.

Kau, J. B., D. Keenan, W. Muller, III, and J. E. Epperson, 1992, A generalized valuation model for fixed-rate residential mortgages, *Journal of Money, Credit and Banking* 24, 279–299.

———, 1993, Option theory and floating-rate securities with a comparison of adjustable and fixed-rate mortgages, *The Journal of Business* 66, 595–618.

McConnell, J. J. and M. K. Singh, 1993, Valuation and analysis of collateralized mortgage obligations, *Management Science* 39, 692–709.

Richard, S. W., 1978, An arbitrate model of the term structure of interest rates, *Journal of Financial Economics* 6, 33–57.

Schwartz, E. S. and W. N. Torous, 1989, Prepayment and the valuation of mortgage-backed securities, *Journal of Finance* 44, 375–392.

Stanton, R., 1993a, Rational prepayment and the valuation of mortgage-backed securities, Unpublished manuscript, University of California, Berkeley.

———, 1993b, Pool heterogeneity and rational learning: The valuation of mortgage-backed securities, Unpublished manuscript, University of California, Berkeley.

Vasicek, O. A., 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177–188.